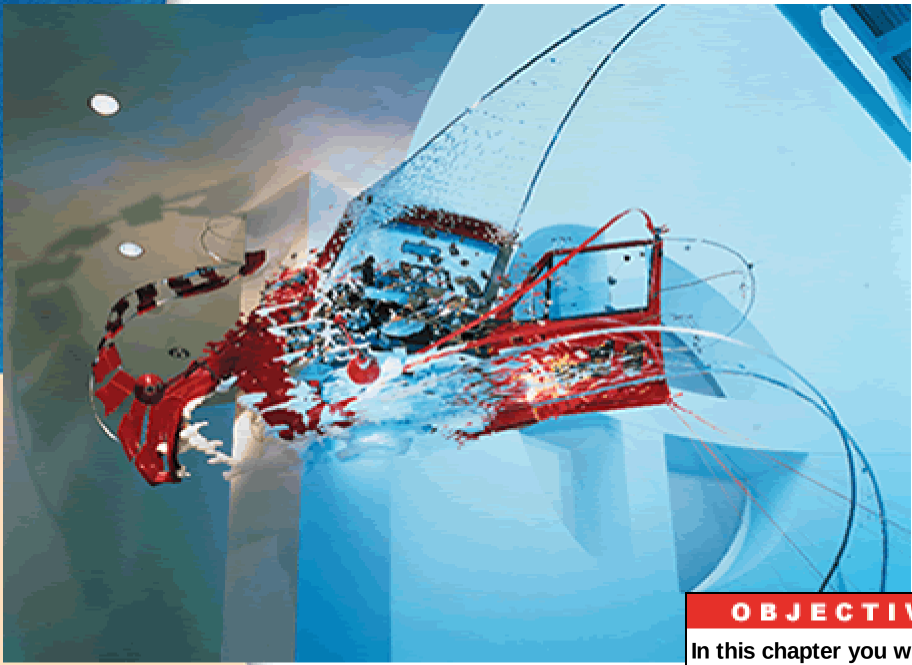
CHAPTER 7

Quadratic and Other Polynomial Functions



American artist Sarah Sze (b 1969) creates flowing sculptures, such as this one created for an exhibit at the San Francisco Museum of Modern Art. This piece features a fractured sport utility vehicle whose pieces have been replaced with disposable household items, including foam packing peanuts. Some of the curves in this artwork's cascade resemble the graphs of polynomial functions.

Things Fall Apart: 2001, mixed media installation with vehicle; variable dimensions/San Francisco Museum of Modern Art, Accessions Committee Fund purchase © Sarah Sze

OBJECTIVES

In this chapter you will

- **find polynomial functions that fit a set of data**
- **study quadratic functions in general form, vertex form, and factored form**
- **find roots of a quadratic equation from a graph, by factoring, and by using the quadratic formula**
- **define complex numbers and operations with them**
- **identify features of the graph of a polynomial function**
- **use division and other strategies to find roots of higher-degree polynomials**

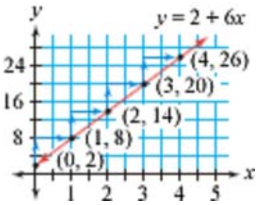
LESSON 7.1

Polynomial Degree and Finite Differences

Differences challenge assumptions.

ANNE WILSON SCHAEF

In Chapter 1, you studied arithmetic sequences, which have a common difference between consecutive terms. If you graph the points of an arithmetic sequence and draw a line through them, this line has a constant slope. So, if you choose *x*-values along the line that form an arithmetic sequence, the corresponding *y*-values will also form an arithmetic sequence.



You have also studied several kinds of nonlinear sequences and functions, which do not have a common difference or a constant slope. In this lesson you will discover that even nonlinear sequences sometimes have a special pattern in their differences.

A **polynomial** expression is a sum of **terms** containing the same variable raised to different powers. Each term is a product of numbers and variables. When a polynomial is set equal to a second variable, such as *y*, you have a **polynomial function.**

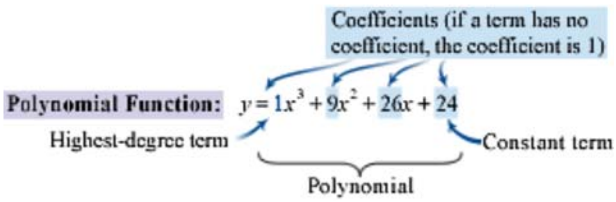
Definition of a Polynomial

A **polynomial** in one variable is any expression that can be written in the form

$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x^1 + a_0$

where *x* is a variable, the exponents are nonnegative integers, and the coefficients are real numbers.

The **degree** of a polynomial or polynomial function is the power of the term that has the greatest exponent. Linear functions are 1st-degree polynomial functions because the largest power of *x* is 1. The polynomial function below has degree 3. If the degrees of the terms of a polynomial decrease from left to right, the polynomial is in **general form.**



A polynomial that has only one term is called a **monomial.** A polynomial with two terms is a **binomial,** and a polynomial with three terms is a **trinomial.** Polynomials with more than three terms are usually just called "polynomials."

In modeling linear functions, you have already discovered that for *x*-values that are uniformly spaced, the differences between the corresponding *y*-values must be the same. With 2nd- and 3rd-degree polynomial functions, the differences between the corresponding y-values are not the same. However, finding the differences between those differences produces an interesting pattern.

1st degree
$y = 3x + 4$

x	y	D_1
2	10	
		3
3	13	
		3
4	16	
		3
5	19	
		3
6	22	
		3
7	25	

2nd degree
$y = 2x^2 - 5x - 7$

x	y	D_1	D_2
3.7	1.88		
		1	
3.8	2.88		0.04
		1.04	
3.9	3.92		0.04
		1.08	
4.0	5.00		0.04
		1.12	
4.1	6.12		0.04
		1.16	
4.2	7.28		

3rd degree
$y = 0.1x^3 - x^2 + 3x - 5$

x	y	D_1	D_2	D_3
–5	–57.5			
		52.5		
0	–5		–50	
		2.5		75
5	–2.5		25	
		27.5		75
10	25		100	
		127.5		75
15	152.5		175	
		302.5		
20	455			

Note that in each case the x-values are spaced equally. You find the first set of differences, D_1, by subtracting each y-value from the one after it. You find the second set of differences, D_2, by finding the differences of consecutive D_1 values in the same way. Notice that for the 2nd-degree polynomial function, the D_2 values are constant, and that for the 3rd-degree polynomial function, the D_3 values are constant. What do you think will happen with a 4th- or 5th-degree polynomial function?

You can use this method to find the degree of the polynomial function that models a certain set of data. Analyzing differences to find a polynomial's degree is called the **finite differences method.**

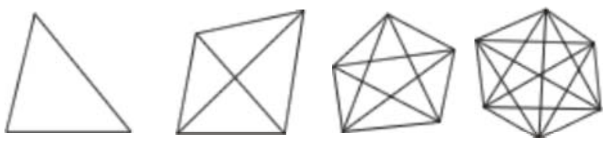
Similar types of foods are grouped together at this market in Camden Lock, London, England. Polynomials are often grouped into similar types as well.

EXAMPLE

Find a polynomial function that models the relationship between the number of sides and the number of diagonals of a polygon. Use the function to find the number of diagonals of a dodecagon (a 12-sided polygon).

▶ Solution

You need to create a table of values with evenly spaced x-values. Sketch polygons with increasing numbers of sides. Then draw all of their diagonals.



Let x be the number of sides and y be the number of diagonals. You may notice a pattern in the number of diagonals that will help you extend your table beyond the sketches you make. Calculate the finite differences to determine the degree of the polynomial function. (Remember that your x-values must be spaced equally in order to use finite differences.)

Number of sides x	Number of diagonals y
3	0
4	2
5	5
6	9
7	14
8	20

D_1	D_2
2	1
3	1
4	1
5	1
6	

You can stop finding differences when the values of a set of differences are constant. Because the values of D_2 are constant, you can model the data with a 2nd-degree polynomial function like $y = ax^2 + bx + c$.

To find the values of a, b, and c, you need a system of three equations. Choose three of the points from your table, say (4, 2), (6, 9), and (8, 20), and substitute the coordinates into $y = ax^2 + bx + c$ to create a system of three equations in three variables. Can you see how these three equations were created?

$$\begin{cases} 16a + 4b + c = 2 \\ 36a + 6b + c = 9 \\ 64a + 8b + c = 20 \end{cases}$$

Solve the system to find $a = 0.5$, $b = -1.5$, and $c = 0$. Use these values to write the function $y = 0.5x^2 - 1.5x$. This equation gives the number of diagonals of any polygon as a function of the number of sides.
Now substitute 12 for x to find that a dodecagon has 54 diagonals.

$$y = 0.5x^2 - 1.5x$$
$$y = 0.5(12)^2 - 1.5(12)$$
$$y = 54$$

With exact function values, you can expect the differences to be equal when you find the right degree. But with experimental or statistical data, as in the investigation, you may have to settle for differences that are nearly constant and that do not show an increasing or decreasing pattern when graphed.

Science CONNECTION

Italian mathematician, physicist, and astronomer Galileo Galilei (1564–1642) performed experiments with free-falling objects. He discovered that the speed of a falling object at any moment is proportional to the amount of time it has been falling. In other words, the longer an object falls, the faster it falls. To learn more about Galileo's experiments and discoveries, see the links at www.keymath.com/DAA.

Investigation

Free Fall

You will need

- a motion sensor
- a small pillow or other soft object

What function models the height of an object falling due to the force of gravity? Use a motion sensor to collect data, and analyze the data to find a function.

Procedure Note

1. Set the sensor to collect distance data approximately every 0.05 s for 2 to 5 s. [► See **Calculator Note 7A** to learn how to set up your calculator. ◄]
2. Place the sensor on the floor. Hold a small pillow at a height of about 2 m, directly above the sensor.
3. Start the sensor and drop the pillow.

Step 1 Follow the procedure note to collect data for a falling object. Let x represent time in seconds, and let y represent height in meters. Select about 10 points from the free-fall portion of your data, with x-values forming an arithmetic sequence. Record this information in a table. Round all table values to the nearest 0.001.

Step 2 Use the finite differences method to find the degree of the polynomial function that models your data. Stop when the differences are nearly constant.

Step 3 Enter your time values, x, into list L1 on your calculator. Enter your height values, y, into list L2. For your first differences, enter your time values without the first value into list L3, and enter the first differences, D_1, into list L4. For your second differences, enter the time values without the first two values into list L5, and enter the second differences, D_2, into list L6. Continue this process for any other differences you calculated. Then make scatter plots of (L1, L2), (L3, L4), (L5, L6), and so on. [► See **Calculator Note 7B** to learn how to calculate finite differences and how to graph them. ◄]

Step 4 Write a description of each graph from Step 3 and what these graphs tell you about the data.

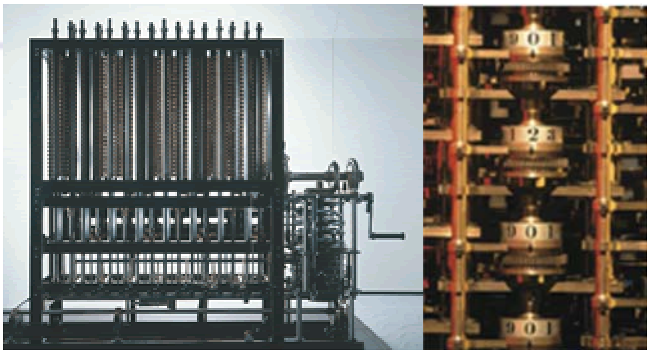
Step 5 Based on your results from using finite differences, what is the degree of the polynomial function that models free fall? Write the general form of this polynomial function.

Step 6 Follow the example on page 362 to write a system of three equations in three variables for your data. Solve your system to find an equation to model the position of a free-falling object dropped from a height of 2 m.

When using experimental data, you must choose your points carefully. When you collect data, as you did in the investigation, your equation will most likely not fit all of the data points exactly due to some errors in measurement and rounding. To minimize the effects of these errors, choose representative points that are not close together, just as you did when fitting a line to data.

History CONNECTION

The method of finite differences was used by the Chinese astronomer Li Shun-Fêng in the 7th century to find a quadratic equation to model the Sun's apparent motion across the sky as a function of time. The Persian astronomer Jamshid Masud al-Kashi, who worked at the Samarkand Observatory in the 15th century, also used the finite differences method when calculating the celestial longitudes of planets. The finite differences method was further developed in 17th- and 18th-century Europe. Scientists used it to eliminate calculations involving multiplication and division when constructing tables of polynomial values. It was not uncommon for a late-18th-century European scientist to have more than 125 volumes of various kinds of tables. In the 19th century, early automatic calculating machines were programmed to calculate differences and were called difference engine



English mathematician and inventor Charles Babbage (1792-1871) designed the first difference engine in the early 1820s, and completed his Difference Engine No. 1, shown here, in 1832

Note that some functions, such as logarithmic and trigonometric functions, cannot be expressed as polynomials. The finite differences method will not produce a set of constant differences for functions other than polynomial functions.

EXERCISES

Practice Your Skills

1. Identify the degree of each polynomial.

a. $x^3 + 9x^2 + 26x + 24$

b. $7x^2 - 5x$

c. $x^7 + 3x^6 - 5x^5 + 24x^4 + 17x^3 - 6x^2 + 2x + 40$

d. $16 - 5x^2 + 9x^5 + 36x^3 + 44x$

2. Determine which of these expressions are polynomials. For each polynomial, state its degree and write it in general form. If it is not a polynomial, explain why not.

a. $-3+4x-3.5x^2+\frac{5}{9}x^3$

b. $5p^4+3.5p-\frac{4}{p^2}+16$

c. $4\sqrt{x^3}+12$

d. $x^2\sqrt{15}-x-4^{-2}$

3. For each data set, decide whether the last column shows constant values. If it does not, calculate the next set of finite differences.

a.

x	*y*
2	4.4
3	6.6
4	9.2
5	11.0
6	10.8
7	7.4

b.

x	*y*	*D*1
3.7	–8.449	
3.8	–8.706	–0.257
3.9	–8.956	–0.250
4.0	–9.200	–0.244
4.1	–9.436	–0.236
4.2	–9.662	–0.227

c.

x	*y*	*D*1	*D*2
–5	–101		
0	–6	95	
5	–11	–5	–100
10	34	45	50
15	279	245	200
20	874	595	350

4. Find the degree of the polynomial function that models these data.

x	0	2	4	6	8	10	12
y	12	– 4	–164	– 612	–1492	–2948	–5124

Keymath.com More Practice Your Skills

Reason and Apply

5. Consider the data at right.

n	1	2	3	4	5	6
s	1	3	6	10	15	21

a. Calculate finite differences to find the degree of the polynomial function that models these data.

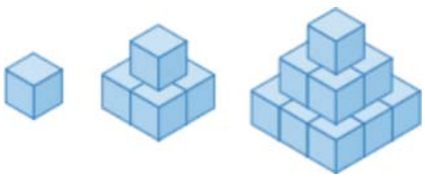
b. Describe how the degree of this polynomial function is related to the finite differences you calculated.

c. What is the minimum number of data points required to determine the degree of this polynomial function? Why?

d. Find the polynomial function that models these data and use it to find *s* when *n* is 12.

e. The values in the *s* row are called triangular numbers. Why do you think they are called triangular? (*Hint:* Find some pennies and try to arrange each number of pennies into a triangle.)

6. You can use blocks to build pyramids such as these. All of the pyramids are solid with no empty space inside.



a. Create a table to record the number of layers, *x*, in each pyramid and the total number of blocks, *y*, needed to build it. You may need to build or sketch a few more pyramids, or look for patterns in the table.

b. Use finite differences to find a polynomial function that models these data.

c. Find the number of blocks needed to build a pyramid with eight layers.

d. Find the number of layers in a pyramid built with 650 blocks.

7. The data in these tables represent the heights of two objects at different times during free fall.

i.

Time (s) t	0	1	2	3	4	5	6
Height (m) h	80	95.1	100.4	95.9	81.6	57.5	23.6

ii.

Time (s) t	0	1	2	3	4	5	6
Height (m) h	4	63.1	112.4	151.9	181.6	201.5	211.6

a. Calculate the finite differences for each table.
b. What is the degree of the polynomial function that you would use to model each data set?
c. Write a polynomial function to model each set of data. Check your answer by substituting one of the data points into your function.

8. Andy has measured his height every three months since he was $9\frac{1}{2}$ years old. Below are his measurements in meters.

Age (yr)	Height (m)	Age (yr)	Height (m)
9.5	1.14	11.5	1.35
9.75	1.21	11.75	1.35
10	1.27	12	1.36
10.25	1.31	12.25	1.37
10.5	1.33	12.5	1.39
10.75	1.34	12.75	1.42
11	1.35	13	1.47
11.25	1.35	13.25	1.54

a. Find the first differences for Andy's heights and make a scatter plot of points in the form (age, D_1). Remember to shorten the list of ages to match D_1. Describe the pattern you see.
b. Repeat your process from 8a until the differences are nearly constant and show no pattern.
c. What type of model will fit these data? Why? Define variables and find a model. For what
d. domain values do you think this model is reasonable?

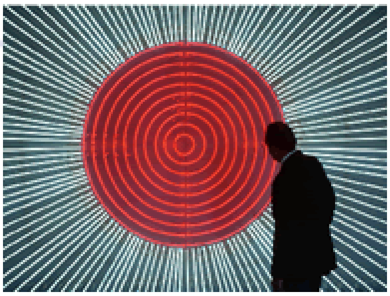
9. **APPLICATION** In an atom, electrons spin rapidly around a nucleus. An electron can occupy only specific energy levels, and each energy level can hold only a certain number of electrons. This table gives the greatest number of electrons that can be in any one level.

Energy level	1	2	3	4	5	6	7
Maximum number of electrons	2	8	18	32	50	72	98

Is it possible to find a polynomial function that expresses the relationship between the energy level and the maximum number of electrons? If so, find the function. If not, explain why not.

Science CONNECTION

The electrons in an atom exist in various energy levels. When an electron moves from a lower energy level to a higher energy level, the atom absorbs energy. When an electron moves from a higher to a lower energy level, energyis released (often as light). This is the principle behind neon lights. The electricity running through a tube of neon gas makes the electrons in the neon atoms jump to higher energy levels. When they drop back to their original level, they give off light.



A visitor admires a neon art display at the Yerba Buena Center for the Arts in San Francisco, California.

Review

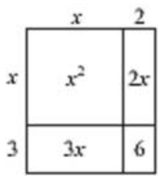
10. Sketch a graph of each function without using your calculator.

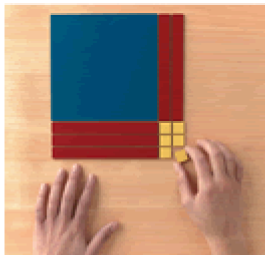
a. $y = (x-2)^2$ **b.** $y = x^2 - 4$ **c.** $y = (x+4)^2 + 1$

11. Solve.

a. $12x - 17 = 13$ **b.** $2(x-1)^2 + 3 = 11$ **c.** $3(5^x) = 48$

12. You may recall that a rectangle diagram can represent the product of two binomials. For example, the rectangle at right represents the product $(x+2)(x-3)$, which you can write as the trinomial $x^2 + 5x + 6$.





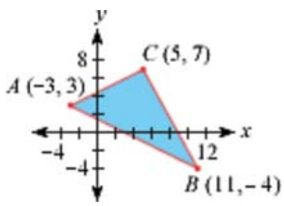
a. Draw a rectangle diagram that represents the product $(2x+3)(3x+1)$.

b. Express the area in 12a as a polynomial in general form.

c. Draw a rectangle whose area represents the polynomial $x^2 + 8x + 15$. (*Hint:* You need to break $8x$ into two terms.)

d. Express the area in 12c as a product of two binomials.

13. Write a system of inequalities that describes the feasible region graphed at right.



14. Find the product $(x+3)(x+4)(x+2)$.

LESSON

7.2

Equivalent Quadratic Forms

I'm very well acquainted, too, with matters mathematical. I understand equations both the simple and quadratical.

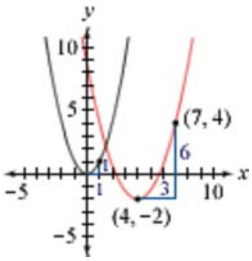
WILLIAM S. GILBERT AND ARTHUR SULLIVAN

This fountain near the Centre Pompidou in Paris, France, contains 16 animated surreal sculptures inspired by the music of Russian-American composer Igor Stravinsky (1882–1971). It was designed by artists Jean Tinguely (1925–1991) and Niki de Saint-Phalle (b 1930). The arc formed by spouting water can be described with a quadratic equation.

In Lesson 7.1, you were introduced to polynomial functions, including 2nd - degree polynomial functions, or **quadratic functions.** The **general form** of a quadratic function is $y = ax^2 + bx + c$. In this lesson you will work with two additional, equivalent forms of quadratic functions.

Recall from Chapter 4 that every quadratic function can be considered as a transformation of the graph of the parent function $y = x^2$. A quadratic function in the form $\frac{y-k}{a} = \left(\frac{x-h}{b}\right)^2$ or $y = a\left(\frac{x-h}{b}\right)^2 + k$ identifies the location of the vertex, (h, k), and the vertical and horizontal scale factors, a and b.

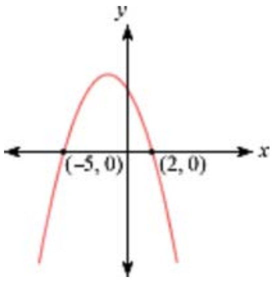
Consider the parabola at right with vertex (4, –2). If you consider the point (7, 4) to be the image of the point (1, 1) on the graph of $y = x^2$, the horizontal scale factor is 3 and the vertical scale factor is 6. So, the quadratic function is $\frac{y+2}{6} = \left(\frac{x-4}{3}\right)^2$ or $y = 6\left(\frac{x-4}{3}\right)^2 - 2$. Choosing a different point as the image of (1, 1) would give an equivalent equation.

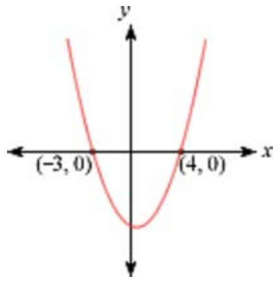


If you move the denominator outside the parentheses, the quadratic function above can also be written as $y = \frac{6}{9}(x-4)^2 - 2$ or $y = \frac{2}{3}(x-4)^2 - 2$. Notice that the horizontal and vertical scale factors are now represented by one vertical scale factor of $\frac{2}{3}$. This coefficient, $\frac{a}{b^2}$, combines the horizontal and vertical scale factors into one vertical scale factor, which you can think of as a single coefficient, say a.

This new form, $y = a(x - h)^2 + k$, is called the **vertex form** of a quadratic function because it identifies the vertex, (h, k), and a single vertical scale factor, a. If you know the vertex of a parabola and one other point, then you can write the quadratic function in vertex form.

Now consider these parabolas. The x-intercepts are marked.





The y-coordinate of any point along the x-axis is 0, so the y-coordinate is 0 at each x-intercept. For this reason, the x-intercepts of the graph of a function are called the **zeros** of the function. You will use this information and the **zero-product property** to find the zeros of a function without graphing.

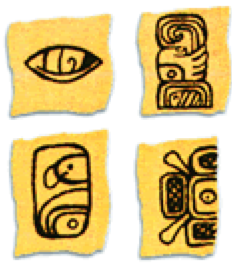
Zero-Product Property

For all real numbers a and b, if $ab = 0$, then $a = 0$, or $b = 0$, or $a = 0$ and $b = 0$.

To understand the zero-product property, think of numbers whose product is zero. Whatever numbers you think of will have this characteristic: *At least one of the factors must be zero.* Before moving on, think about numbers that satisfy each equation below.

$\underline{\ ?\ } \cdot 16.2 = 0$

$3(\underline{\ ?\ } - 4)(\underline{\ ?\ } - 9) = 0$



These Mayan representations of zero were used as placeholders, as in "100," rather than to symbolize "nothingness."

EXAMPLE A Find the zeros of the function $y = -1.4(x - 5.6)(x + 3.1)$.

Solution The zeros will be the x-values that make y equal 0. First, set the function equal to zero.

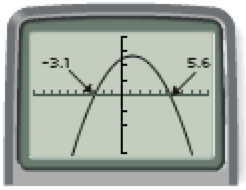
$$0 = -1.4(x - 5.6)(x + 3.1)$$

Because the product of three factors equals zero, the zero-product property tells you that at least one of the factors must equal zero.

$-1.4 = 0$	or	$x - 5.6 = 0$	or	$x + 3.1 = 0$
not possible		$x = 5.6$		$x = -3.1$

So the solutions, or **roots,** of the equation $0 = -1.4(x - 5.6)(x + 3.1)$ are $x = 5.6$ or $x = -3.1$. That means the zeros of the function $y = -1.4(x - 5.6)(x + 3.1)$ are $x = 5.6$ and $x = -3.1$.

Use your graphing calculator to check your work. You should find that the x-intercepts of the graph of $y = -1.4(x - 5.6)(x + 3.1)$ are 5.6 and –3.1.

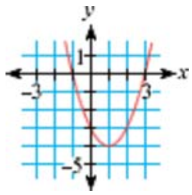


[-10, 10, 1, -40, 40, 10]

If you know the x-intercepts of a parabola, then you can write the quadratic function in **factored form,** $y = a(x - r_1)(x - r_2)$. This form identifies the locations of the x-intercepts, r_1 and r_2, and a vertical scale factor, a.

EXAMPLE B

Consider the parabola at right.



a. Write an equation of the parabola in vertex form.

b. Write an equation of the parabola in factored form.

c. Show that both equations are equivalent by converting them to general form.

▶ Solution

The vertex is (1, –4). If you consider the point (2, –3) to be the image of the point (1, 1) on the graph of $y = x^2$, then the vertical and horizontal scale factors are both 1. So, the single vertical scale factor is $a = \frac{1}{1^2} = 1$.

a. The vertex form is

$$y = (x - 1)^2 - 4$$

b. The x-intercepts are –1 and 3. You know the scale factor, a, is 1. So the factored form is

$$y = (x + 1)(x - 3)$$

c. To convert to general form, multiply the binomials and then combine like terms. The use of rectangle diagrams may help you multiply the binomials.

$y = (x - 1)^2 - 4$
$y = (x - 1)(x - 1) - 4$
$y = (x^2 - x - x + 1) - 4$
$y = x^2 - 2x - 3$

	x	-1
x	x^2	$-x$
-1	$-x$	1

$y = (x + 1)(x - 3)$
$y = x^2 + x - 3x - 3)$
$y = x^2 - 2x - 3$

	x	1
x	x^2	x
-3	$-3x$	-3

The vertex form and the factored form are equivalent because they are both equivalent to the same general form.

You now know three different forms of a quadratic function.

Three Forms of a Quadratic Function

General form	$y = ax^2 + bx + c$
Vertex form	$y = a(x - h)^2 + k$
Factored form	$y = a(x - r_1)(x - r_2)$

The investigation will give you practice in using the three forms with real data. You'll find that the form you use guides which features of the data you focus on. Conversely, if you know only a few features of the data, you may need to focus on a particular form of the function.

This painting by Elizabeth Catlett, which depicts people singing songs in different forms, was inspired by a poem titled "For My People" by American writer Margaret Walker Alexander (1915-1998).

Elizabeth Catlett (American, b 1915), *Singing Their Songs* (1992) Lithograph on paper (a.p.#6) 15-3/4 x 13-3/4 in. / National Museum of Women in the Arts, purchased with funds donated in memory of Florence Davis by her family, friends, and the NMWA Women's Committee

Investigation
Rolling Along

You will need

- a motion sensor
- an empty coffee can
- a long table

Procedure Note

Prop up one end of the table slightly. Place the motion sensor at the low end of the table and aim it toward the high end. With tape or chalk, mark a starting line 0.5 m from the sensor on the table.

Step 1 Practice rolling the can up the table directly in front of the motion sensor. Start the can behind the starting line. Give the can a gentle push so that it rolls up the table on its own momentum, stops near the end of the table, and then rolls back. Stop the can after it crosses the line and before it hits the motion sensor.

Step 2 Set up your calculator to collect data for 6 seconds. [See Calculator Note 7C.] When the sensor begins, roll the can up the table.

Step 3 The data collected by the sensor will have the form (*time*, *distance*). Adjust for the position of the starting line by subtracting 0.5 from each value in the distance list.

Step 4 Let x represent time in seconds, and let y represent distance from the line in meters. Draw a graph of your data. What shape is the graph of the data points? What type of function would model the data? Use finite differences to justify your answer.

Step 5 Mark the vertex and another point on your graph. Approximate the coordinates of these points and use them to write the equation of a quadratic model in vertex form.

Step 6 From your data, find the distance of the can at 1, 3, and 5 seconds. Use these three d ata points to find a quadratic model in general form.

Step 7 Mark the x-intercepts on your graph. Approximate the values of these x-intercepts. Use the zeros and the value of a from Step 5 to find a quadratic model in factored form.

Step 8 Verify by graphing that the three equations in Steps 5, 6, and 7 are equivalent, or nearly so. Write a few sentences explaining when you would use each of the three forms to find a quadratic model to fit parabolic data.

You can find a model for data in different ways, depending on the information you have. Conversely, different forms of the same equation give you different kinds of information. Being able to convert one form to another allows you to compare equations written in different forms. In the exercises, you will convert both the vertex form and the factored form to the general form. In later lessons you will learn other conversions.

EXERCISES

Practice Your Skills

For the exercises in this lesson, you may find it helpful to use a window on your calculator that has friendly x-values. A background grid may also be helpful.

1. Identify each quadratic function as being in general form, vertex form, factored form, or none of these forms.

a. $y = -3.2(x + 4.5)^2$ **b.** $y = 2.5(x + 1.25)(x - 1.25) + 4$

c. $y = 2x(3 + x)$ **d.** $y = 2x^2 - 4.2x - 10$

2. Each quadratic function below is written in vertex form. What are the coordinates of each vertex? Graph each equation to check your answers.

a. $y = (x - 2)^2 + 3$ **b.** $y = 0.5(x + 4)^2 - 2$ **c.** $y = 4 - 2(x - 5)^2$

3. Each quadratic function below is written in factored form. What are the zeros of each function? Graph each equation to check your answers.

a. $y = (x + 1)(x - 2)$ **b.** $y = 0.5(x - 2)(x + 3)$ **c.** $y = -2(x - 2)(x - 5)$

4. Convert each function to general form. Graph both forms to check that the equations are equivalent.

a. $y = (x - 2)^2 + 3$ **b.** $y = 0.5(x + 4)^2 - 2$ **c.** $y = 4 - 2(x - 5)^2$

5. Convert each function to general form. Graph both forms to check that the equations are equivalent.

a. $y = (x + 1)(x - 2)$ **b.** $y = 0.5(x - 2)(x + 3)$ **c.** $y = -2(x - 2)(x - 5)$

When you see an image in a different form, your attention is drawn to different features.

More Practice Your Skills

Reason and Apply

6. As you learned in Chapter 4, the graphs of all quadratic functions have a line of symmetry that contains the vertex and divides the parabola into mirror-image halves. Consider this table of values generated by a quadratic function.

x	y
1.5	–8
2.5	7
3.5	16
4.5	19
5.5	16
6.5	7
7.5	–8

a. What is the line of symmetry for the graph of this quadratic function?

b. The vertex of a parabola represents either the **maximum** or **minimum** value of the quadratic function. Name the vertex of this function and determine whether it is a maximum or minimum.

c. Use the table of values to write the quadratic function in vertex form.

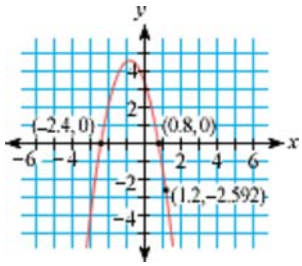
7. Write each function in general form.

a. $y = 4 - 0.5(x + h)^2$ **b.** $y = a(x - 4)^2$

c. $y = a(x - h)^2 + k$ **d.** $y = -0.5(x + r)(x + 4)$

e. $y = a(x - 4)(x + 2)$ **f.** $y = a(x - r)(x - s)$

8. At right is the graph of the quadratic function that passes through (–2.4, 0), (0.8, 0), and (1.2, –2.592).



a. Use the x-intercepts to write the quadratic function in factored form. For now, leave the vertical scale factor as a.

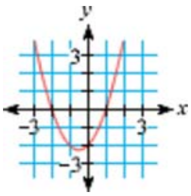
b. Substitute the coordinates of (1.2, –2.592) into your function from 8a, and solve for a. Write the complete quadratic function in factored form.

c. The line of symmetry for the graph of this quadratic function passes through the vertex and the point on the x-axis halfway between the two x-intercepts. What is the x-coordinate of the vertex? What is the y-coordinate?

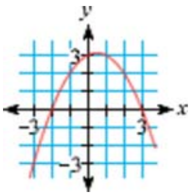
d. Write this quadratic function in vertex form.

9. Write the factored form for each polynomial function. (*Hint:* Substitute the coordinates of the y-intercept to solve for the scale factor, a.)

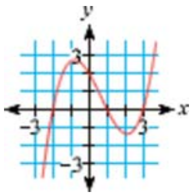
a.



b.



c.



10. **APPLICATION** A local outlet store charges \$2.00 for a pack of four AA batteries. On an average day, 200 packs are sold. A survey indicates that sales will decrease by 5 packs per day for each \$0.10 increase in price.

Selling price (\$)	2.00	2.10	2.20	2.30	2.40
Number sold	200	?	?	?	?
Revenue (\$)	400	?	?	?	?

a. Complete the table above based on the results of the survey.

b. Calculate the first and second differences for the revenue.

c. Let x represent the selling price in dollars, and let y represent the revenue in dollars. Write a function that describes the relationship between the revenue and the selling price.

d. Graph your function and find the maximum revenue. What selling price provides maximum revenue?

11. **APPLICATION** Delores has 80 m of fence to surround an area where she is going to plant a vegetable garden. She wants to enclose the largest possible rectangular area.

Width (m)	5	10	15	20	25
Length (m)	?	?	?	?	?
Area (m^2)	?	?	?	?	?

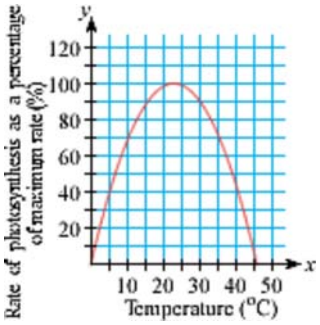
a. Copy and complete this table.

b. Let x represent the width in meters, and let y represent the area in square meters. Write a function that describes the relationship between the area and the width of the garden.

c. Which width provides the largest possible area? What is that area?

d. Which widths result in an area of 0 m^2?

12. **APPLICATION** Photosynthesis is the process in which plants use energy from the sun, together with CO_2 (carbon dioxide) and water, to make their own food and produce oxygen. Various factors affect the rate of photosynthesis, such as light intensity, light wavelength, CO_2 concentration, and temperature. Below is a graph of how temperature relates to the rate of photosynthesis for a particular plant. (All other factors are assumed to be held constant.)



When chlorophyll fades for the winter, leaves change color. When they appear red or purple, it is because glucose is trapped in the leaves and changes color when exposed to sunlight and cool nights.

a. Describe the general shape of the graph. What does the shape of the graph mean in the context of photosynthesis?

b. Approximate the optimum temperature for photosynthesis in this plant and the corresponding rate of photosynthesis.

c. Temperature has to be kept within a certain range for photosynthesis to occur. If it gets too hot, then the enzymes in chlorophyll are killed and photosynthesis stops. If the temperature is too cold, then the enzymes stop working. At approximately what temperatures is the rate of photosynthesis equal to zero?

d. Write a function in at least two forms that will produce this graph.

Review

13. Use a rectangle diagram to find each product.

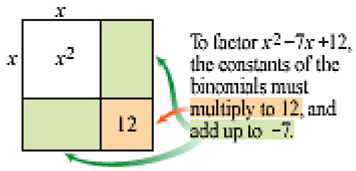
a. $3x(4x - 5)$ **b.** $(x + 3)(x - 5)$ **c.** $(x + 7)(x - 7)$ **d.** $(3x - 1)^2$

14. Recall that the distributive property allows you to distribute a factor through parentheses. The factor that is distributed doesn't have to be a monomial. Here's an example with a binomial.

$$(x+1)(x-3)$$
$$x(x-3)+1(x-3)$$
$$x^2-3x+x-3$$
$$x^2-2x-3$$

Use the distributive property to find each product in Exercise 13.

15. You can also use a rectangle diagram to help you **factor** some trinomials, such as $x^2 - 7x + 12$.



	x	-3
x	x^2	$-3x$
-4	$-4x$	12

$(x-3)(x-4)$ The constants of the binomials need to be –3 and –4.

Use rectangle diagrams to help you factor these trinomials.

a. $x^2 + 3x - 10$ **b.** $x^2 + 8x + 16$ **c.** $x^2 - 25$

16. Use the function $f(x) = 3x^3 - 5x^2 + x - 6$ to find these values.

a. $f(2)$ **b.** $f(-1)$ **c.** $f(0)$ **d.** $f\left(\frac{1}{2}\right)$ **e.** $f\left(-\frac{4}{3}\right)$

IMPROVING YOUR REASONING SKILLS

Sums and Differences

Use the method of finite differences to find a formula for the sum of the first n terms in the arithmetic sequence 1, 2, 3, 4, ... In other words, find a formula for the sum

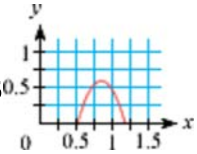
$$1 + 2 + 3 + \cdots + u_n$$

Then find a formula for the first n terms of the sequence 1, 3, 5, 7, ... How about 1, 4, 7, 10, . . . ? Look for patterns, and see if you can write a formula that will determine the sum of the first n terms of any arithmetic sequence with a first term of 1, and common difference d.

LESSON 7.3

Completing the Square

The graph of $y = -5.33(x - 0.86)^2 + 0.6$ at right models one bounce of a ball, where x is time in seconds and y is height in meters. The maximum height of this ball occurs at the vertex (0.86, 0.6), which means that after 0.86 s the ball reaches its maximum height of 0.6 m. Finding the **maximum** or **minimum** value of a quadratic function is often necessary to answer questions about data. Finding the vertex is straightforward when you are given an equation in vertex form and sometimes when you are using a graph. However, you often have to estimate values on a graph. In this lesson you will learn a procedure called **completing the square** to convert a quadratic equation from general form to vertex form accurately.



An object that rises and falls under the influence of gravity is called a projectile. You can use quadratic functions to model **projectile motion,** or the height of the object as a function of time.

The height of a projectile depends on three things: the height from which it is thrown, the upward velocity with which it is thrown, and the effect of gravity pulling downward on the object. So, the polynomial function that describes projectile motion has three terms. The leading coefficient of the polynomial is based on the acceleration due to gravity, g. On Earth, g has an approximate numerical value of 9.8 m/s^2 when height is measured in meters and 32 ft/s^2 when height is measured in feet. The leading coefficient of a projectile motion function is always $-\frac{1}{2}g$.

Projectile Motion Function

The height of an object rising or falling under the influence of gravity is modeled by the function

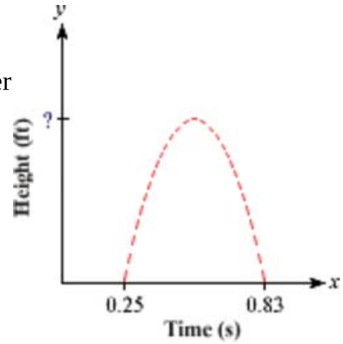
$$y = ax^2 + v_0x + s_0$$

where x represents time in seconds, y represents the object's height from the ground in meters or feet, a is half the downward acceleration due to gravity (on Earth, a is -4.9 m/s^2 or -16 ft/s^2), v_0 is the initial upward velocity of the object in meters per second or feet per second, and s_0 is the initial height of the object in meters or feet.

The water erupting from these geysers in Black Rock Desert, Nevada, follows a path that can be described as projectile motion.

EXAMPLE A

A stopwatch records that when Julie jumps in the air, she leaves the ground at 0.25 s and lands at 0.83 s. How high was her jump, in feet?



▶ Solution

You don't know the initial velocity, so you can't yet use the projectile motion function. But you do know that height is modeled by a quadratic function and that the leading coefficient must be approximately –16 when using units of feet. Use this information along with 0.25 and 0.83 as the x-intercepts (when Julie's jump height is 0) to write the function

$$y = -16(x - 0.25)(x - 0.83)$$

The vertex of the graph of this equation represents Julie's maximum jump height. The x-coordinate of the vertex will be midway between the two x-intercepts, 0.25 and 0.83. The mean of 0.25 and 0.83 is $\frac{0.25 + 0.83}{2}$, or 0.54.

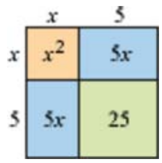
$y = -16(x - 0.25)(x - 0.83)$	The original function.
$y = -16(0.54 - 0.25)(0.54 - 0.83)$	Substitute the x-coordinate of the vertex.
$y \approx 1.35$	Evaluate.

Julie jumped 1.35 ft, or about 16 in.

Example A showed how to find the vertex of a quadratic function in factored form. If the equation were in general form instead, you might be able to put it in factored form and then follow the same steps, but there's another way. First, note that each rectangle diagram below represents a **perfect square** because both factors are the same.

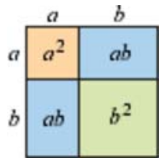
$$(x + 5)^2 = (x + 5)(x + 5)$$
$$= x^2 + 5x + 5x + 25$$
$$= x^2 + 10x + 25$$

The square of the first term of the binomial. (x^2)
Twice the product of the first and second terms of the binomial. ($10x$)
The square of the second term of the binomial. (25)



$$(a + b)^2 = (a + b)(a + b)$$
$$= a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

The square of the first term of the binomial. (a^2)
Twice the product of the first and second terms of the binomial. ($2ab$)
The square of the second term of the binomial. (b^2)



Notice the pattern. For a perfect square, the first and last terms of the trinomial are squares, and the middle term is twice a product. This knowledge will be useful in the investigation, which helps you convert a quadratic function from general form to vertex form.

Investigation

Complete the Square

You can use rectangle diagrams to help convert from general form to vertex form.

Step 1 Consider the expression $x^2 + 6x$.

	x	3
x	x^2	$3x$
3	$3x$	?

a. What could you add to the expression to make it a perfect square? That is, what must you add to complete this rectangle diagram?

b. If you add a number to an expression, then you must also subtract the same amount in order to preserve the value of the original expression. Fill in the blanks to

rewrite $x^2 + 6x$ as the difference between a perfect square and a number.

$x^2 + 6x = x^2 + 6x + \underline{?} - \underline{?} = (x + 3)^2 - \underline{?}$

c. Use a graph or table to verify that your expression in the form $(x - h)^2 + k$ is equivalent to the original expression, $x^2 + 6x$.

Step 2 Consider the expression $x^2 + 6x - 4$.

	x	3
x	x^2	$3x$
3	$3x$	?

a. Focus on the 2nd- and 1st-degree terms of the expression, $x^2 + 6x$. What must be added to and subtracted from these terms to complete a perfect square yet preserve the value of the expression?

b. Rewrite the expression $x^2 + 6x - 4$ in the form $(x - h)^2 + k$.

c. Use a graph or table to verify that your expression is equivalent to the original expression, $x^2 + 6x - 4$.

Step 3 Rewrite each expression in the form $(x - h)^2 + k$. If you use a rectangle diagram, focus on the 2nd- and 1st-degree terms first. Verify that your expression is equivalent to the original expression.

a. $x^2 - 14x + 3$ **b.** $x^2 - bx + 10$

When the 2nd-degree term has a coefficient, you can first factor it out of the 2nd- and 1st-degree terms. For example, $3x^2 + 24x + 5$ can be written $3(x^2 + 8x) + 5$. Completing a diagram for $x^2 + 8x$ can help you rewrite the expression in the form $a(x - h)^2 + k$.

	x	4
x	x^2	$4x$
4	$4x$	16

$3x^2 + 24x + 5$	The original expression.
$3(x^2 + 8x) + 5$	Factor the 2nd- and 1st-degree terms.
$3(x^2 + 8x + 16) - 3(16) + 5$	Complete the square. You add $3 \cdot 16$, so you must subtract $3 \cdot 16$.
$3(x + 4)^2 - 43$	An equivalent expression in the form $a(x - h)^2 + k$.

Step 4 Rewrite each expression in the form $a(x-h)^2+k$. Use a graph or table to verify that your expression is equivalent to the original expression.

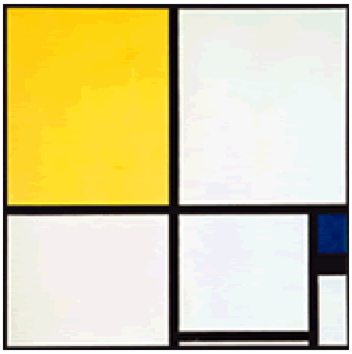
a. $2x^2-6x+1$ **b.** $ax^2+10x+7$

Step 5 Use the strategy from Step 4 to rewrite this expression in the form $a(x-h)^2+k$:

$$ax^2+bx+c$$

Step 6 If you graph the quadratic function $y=ax^2+bx+c$, what will be the coordinates of the vertex in terms of a, b, and c? Why?

In the investigation you saw how to convert a quadratic expression from the form ax^2+bx+c to the form $a(x-h)^2+k$. This process is called **completing the square**. As you saw in Steps 5 and 6, the two forms are related through the relationships $h=-\frac{b}{2a}$ and $k=c-\frac{b^2}{4a}$. You can use these relationships to quickly calculate the coordinates of the vertex of a quadratic function in general form.



Composition with Blue and Yellow (1931), was created by Dutch abstract painter Piet Mondrian (1872–1944).

EXAMPLE B

Convert each quadratic function to vertex form. Identify the vertex.

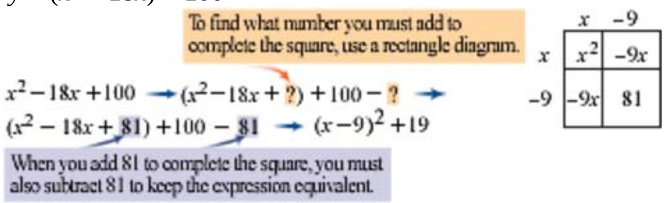
a. $y=x^2-18x+100$ **b.** $y=3x^2+21x-35$

▶ Solution

To convert to vertex form, you must complete the square.

a. First separate the constant from the first two terms.

$y=(x^2-18x)+100$ Original equation.



$y=(x^2-18x+81)+100-81$ Add 81 to complete the square, and subtract 81 to keep the equation equivalent.

$y=(x-9)^2+19$ Factor into a perfect square, and add the constant terms.

The vertex form is $y=(x-9)^2+19$, and the vertex is (9, 19).

b. As an alternative to completing the square for $y = 3x^2 + 21x - 35$, you can use the formulas for h and k. First, identify the coefficients, a, b, and c.

$$y = 3x^2 + 21x - 35$$

$$a = 3 \quad b = 21 \quad c = -35$$

Substitute the values of a, b, and c into the formulas for h and k.

$$h = -\frac{b}{2a} \qquad k = c - \frac{b^2}{4a}$$

$$h = -\frac{21}{2(3)} = -3.5 \qquad k = -35 - \frac{21^2}{4(3)} = -71.75$$

The vertex is $(-3.5, -71.75)$, and the vertex form is $y = 3(x + 3.5)^2 - 71.75$. Remember to include the value of a in the vertex form.

You may find that using the formulas for h and k is often simpler than completing the square. Both methods will allow you to find the vertex of a quadratic equation and write the equation in vertex form. However, the procedure of completing the square will be used again in your work with ellipses and other geometric shapes, so you should become comfortable using it.

EXAMPLE C

Nora hits a softball straight up at a speed of 120 ft/s. If her bat contacts the ball at a height of 3 ft above the ground, how high does the ball travel? When does the ball reach its maximum height?

1999 U.S. Olympic Softball Team player Kim Maher at bat

▸ Solution

Using the projectile motion function, you know that the height of the object at time x is represented by the equation $y = ax^2 + v_0x + s_0$. The initial velocity, v_0, is 120 ft/s, and the initial height, s_0, is 3 ft. Because the distance is measured in feet, the approximate leading coefficient is –16. Thus, the function is $y = -16x^2 + 120x + 3$. To find the maximum height, locate the vertex.

$$y = -16x^2 + 120x + 3$$ Original equation. Identify the coefficients, $a = -16$, $b = 120$, and $c = 3$.

$$h = -\frac{b}{2a} = -\frac{120}{2(-16)} = 3.75$$ Use the formula for h to find the x-coordinate of the vertex.

$$k = c - \frac{b^2}{4a} = 3 - \frac{120^2}{4(-16)} = 228$$ Use the formula for k to find the y-coordinate of the vertex.

The softball reaches a maximum height of 228 ft at 3.75 s.

You now have several strategies for finding the vertex of a quadratic function. You can convert from general form to vertex form by completing the square or by using the formulas for h and k.

EXERCISES

Practice Your Skills

1. Factor each quadratic expression.
   - a. $x^2 - 10x + 25$
   - b. $x^2 + 5x + \frac{25}{4}$
   - c. $4x^2 - 12x + 9$
   - d. $x^2 - 2xy + y^2$

2. What value is required to complete the square?
   - a. $x^2 + 20x + \underline{\ ?\ }$
   - b. $x^2 - 7x + \underline{\ ?\ }$
   - c. $4x^2 - 16x + \underline{\ ?\ }$
   - d. $-3x^2 - 6x + \underline{\ ?\ }$

3. Convert each quadratic function to vertex form.
   - a. $y = x^2 + 20x + 94$
   - b. $y = x^2 - 7x + 16$
   - c. $y = 6x^2 - 24x + 147$
   - d. $y = 5x^2 + 8x$

4. Rewrite each expression in the form $ax^2 + bx + c$, and then identify the coefficients, a, b, and c.
   - a. $3x^2 + 2x - 5$
   - b. $14 + 2x^2$
   - c. $-3 + 4x^2 - 2x + 8x$
   - d. $3x - x^2$

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Hungarian modern sculptor Màrton Vàrò (b 1943) titled this piece *15 Cubes* (1998). It is located at the Elektro building in Trondheim, Norway.

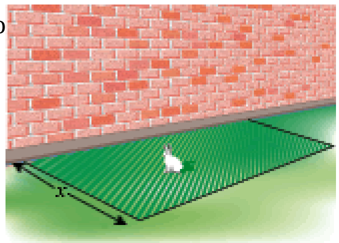
Reason and Apply

5. What is the vertex of the graph of the quadratic function $y = -2x^2 - 16x - 20$?

6. Convert the function $y = 7.51x^2 - 47.32x + 129.47$ to vertex form. Use a graph or table to verify that the functions are equivalent.

7. Imagine that an arrow is shot from the bottom of a well. It passes ground level at 1.1 s and lands on the ground at 4.7 s.
   - a. Define variables and write a quadratic function that describes the height of the arrow, in meters, as a function of time.
   - b. What was the initial velocity of the arrow in meters per second?
   - c. How deep was the well in meters?

8. **APPLICATION** Suppose you are enclosing a rectangular area to create a rabbit cage. You have 80 ft of fence and want to build a pen with the largest possible area for your rabbit, so you build the cage using an existing building as one side.
   - a. Make a table showing the areas for some selected values of x, and write a function that gives the area, y, as a function of the width, x.
   - b. What width maximizes the area? What is the maximum area?



9. A rock is thrown upward from the edge of a 50 m cliff overlooking Lake Superior, with an initial velocity of 17.2 m/s. Define variables and write an equation that models the height of the rock.

10. An object is projected upward, and these data are collected.

Time (s) *t*	1	2	3	4	5	6
Height (m) *h*	120.1	205.4	280.9	346.6	402.4	448.4

a. Write a function that relates time and height for this object.

b. What was the initial height? The initial velocity?

c. When does the object reach its maximum height? What is the maximum height?

11. **APPLICATION** The members of the Young Entrepreneurs Club decide to sell T-shirts in their school colors for Spirit Week. In a marketing survey, the members ask students whether or not they would buy a T-shirt for a specific price. In analyzing the data, club members find that at a price of $20 they would sell 60 T-shirts. For each $5 increase in price, they would sell 10 fewer T-shirts.

a. Find a linear function that relates the price in dollars, p, and the number of T-shirts sold, n.

b. Write a function that gives revenue as a function of price. (Use your function in 11a as a substitute for the number of T-shirts sold.)

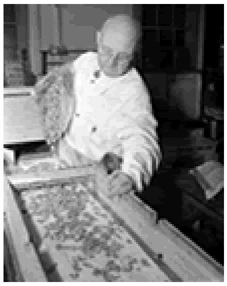
c. Convert the revenue function to vertex form. What is the real-world meaning of the vertex?

d. If the club members want to receive at least $1050 in revenue, what price should they charge for the T-shirts?

Business CONNECTION

Entrepreneurs are individuals who take a risk to create a new product or a new business. Some famous American entrepreneurs include Sarah Breedlove ("Madam C. J. Walker") (1867–1919), who was a pioneer in the cosmetics industry for African-Americans; Clarence Birdseye (1886–1956), who made frozen food available; Ray Kroc (1902–1984), who developed a way to provide fast service and created McDonald's; and Jeff Bezos (b 1964), who founded Amazon.com and popularized a way to sell merchandise without the traditional retail stores.

Sarah Breedlove Walker expanded her cosmetics company across the United States, the Caribbean, and Europe. The Granger Collection, New York City



Around the time of World War II (1939–1945), Clarence Birdseye, shown here dehydrating chopped carrots, created foods that were fast and easy to prepare.

Review

12. Multiply.

a. $(x-3)(2x+4)$

b. $(x^2+1)(x+2)$

13. Solve $(x-2)(x+3)(2x-1)=0$.

14. Consider a graph of the unit circle, $x^2+y^2=1$. Stretch it vertically by a scale factor of 3, and translate it left 5 units and up 7 units.

a. Write the equation of this new shape. What is it called?

b. Sketch a graph of this shape. Label the center and at least four points.

15. **APPLICATION** This table shows the number of endangered species in the United States for selected years from 1980 to 2000.

Endangered Species

Year	1980	1985	1990	1995	1996	1997	1998	1999	2000
Number of endangered species	224	300	442	756	837	896	924	939	961

(*The New York Times Almanac 2002*)

a. Define variables and create a scatter plot of these data.

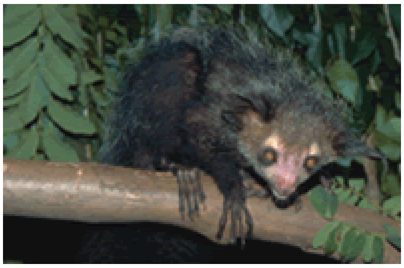
b. Find the equation of the median-median line for these data.

c. Use the median-median line to predict the number of endangered species in 2005 and in 2050.

Environmental CONNECTION

Most species become endangered when humans damage their ecosystems through pollution, habitat destruction, and introduction of nonnative species. Over-hunting and over-collecting also threaten animal and plant populations. Growing human and livestock populations have made this a constantly increasing problem—the current global extinction rate is about 20,000 species a year. For information about what is being done to protect endangered species, see the weblinks at www.keymath.com/DAA .

The Karner blue butterfly, native to the Great Lakes region, has become endangered as the availability of its primary food, blue lupine, has become scarce due to development and fire suppression.



The aye-aye is a nocturnal primate native to Madagascar. Their numbers have declined due to habitat destruction, and because they are considered to be a bad omen, they are often killed. There are estimated to be only 100 left.

LESSON 7.4

The Quadratic Formula

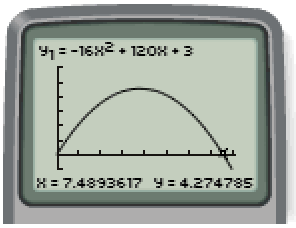
Although you can always use a graph of a quadratic function to approximate the x-intercepts, you are often not able to find exact solutions. This lesson will develop a procedure to find the exact roots of a quadratic equation by first converting the equation to vertex form. Consider again this situation from Example C in the last lesson.

EXAMPLE A

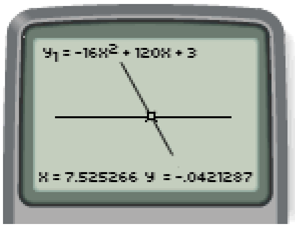
Nora hits a softball straight up at a speed of 120 ft/s. Her bat contacts the ball at a height of 3 ft above the ground. Recall that the equation relating height in meters, y, and time in seconds, x, is $y = -16x^2 + 120x + 3$. How long will it be until the ball hits the ground?

Solution

The height will be zero when the ball hits the ground, so you want to find the solutions to the equation $-16x^2 + 120x + 3 = 0$. You can approximate the x-intercepts by graphing, but as you can see it's difficult to be accurate. Even after you zoom in a few times, you may not be able to find the exact x-intercept.



[0, 8, 1, –50, 300, 50]



[7.46, 7.58, 1, –2.28, 3.18, 50]

You will not be able to factor this equation using a rectangle diagram, so you can't use the zero-product property. Instead, to solve this equation symbolically, first write the equation in the form $a(x - h)^2 + k = 0$.

$-16x^2 + 120x + 3 = 0$	Original equation.
$h = -\frac{b}{2a} = -\frac{120}{2(-16)} = 3.75$	Find the values of h and k.
$k = c - \frac{b^2}{4a} = 3 - \frac{120^2}{4(-16)} = 228$	
$-16(x - 3.75)^2 + 228 = 0$	Use the values of h and k to convert to the form $a(x - h)^2 + k = 0$.
$-16(x - 3.75)^2 = -228$	Subtract 228 from both sides.
$(x - 3.75)^2 = 14.25$	Divide by –16.
$x - 3.75 = \pm\sqrt{14.25}$	Take the square root of both sides.
$x = 3.75 \pm \sqrt{14.25}$	Add 3.75 to both sides.
$x = 3.75 + \sqrt{14.25}$ or $x = 3.75 - \sqrt{14.25}$	Write the two exact solutions to the equation.
$x \approx 7.525$ or $x \approx -0.025$	Approximate the values of x.

The zeros of the function are $x \approx 7.525$ and $x \approx -0.025$. The negative time, –0.025 s, does not make sense in this situation, so the ball hits the ground after approximately 7.525 s.

If you follow the same steps with a general quadratic equation, then you can develop the **quadratic formula.** This formula provides solutions to $ax^2 + bx + c = 0$ in terms of a, b, and c.

$ax^2 + bx + c = 0$	Original equation.
$h = \frac{-b}{2a}$ and $k = c - \frac{b^2}{4a}$	Find the values of h and k.
$a\left(x + \frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$	Rewrite the equation in the form $a(x - h)^2 + k = 0$.
$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$	Subtract c from both sides. Add $\frac{b^2}{4a}$ to both sides.
$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a} - \frac{4ac}{4a}$	Rewrite the right side with a common denominator.
$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a}$	Add terms with a common denominator.
$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$	Divide both sides by a.
$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$	Take the square root of both sides.
$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}$	Use the power of a quotient property to take the square roots of the numerator and denominator.
$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$	Subtract $\frac{b}{2a}$ from both sides.
$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$	Add terms with a common denominator.

The Quadratic Formula

Given a quadratic equation written in the form $ax^2 + bx + c = 0$, the solutions are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

To use the quadratic formula on the equation in Example A, $-16x^2 + 120x + 3 = 0$, first identify the coefficients as $a = -16$, $b = 120$, and $c = 3$. The solutions are

$$x = \frac{-120 \pm \sqrt{120^2 - 4(-16)(3)}}{2(-16)}$$

$x = \frac{-120 + \sqrt{14592}}{-32}$ or $x = \frac{-120 + \sqrt{14592}}{-32}$

$x \approx -0.025$ or $x \approx 7.525$

The quadratic formula gives you a way to find the roots of any equation in the form $ax^2 + bx + c = 0$. The investigation will give you an opportunity to apply the quadratic formula in different situations.

Investigation
How High Can You Go?

Salvador hits a baseball at a height of 3 ft and with an initial upward velocity of 88 feet per second.

Step 1 Let x represent time in seconds after the ball is hit, and let y represent the height of the ball in feet. Write an equation that gives the height as a function of time.

Step 2 Write an equation to find the times when the ball is 24 ft above the ground.

Step 3 Rewrite your equation from Step 2 in the form $ax^2 + bx + c = 0$, then use the quadratic formula to solve. What is the real-world meaning of each of your solutions? Why are there two solutions?

Step 4 The vertex of this parabola has a y-coordinate of 124. Explain why the ball will reach a height of 124 ft only once.

Step 5 Write an equation to find the time when the ball reaches a height of 124 ft. Use the quadratic formula to solve the equation. At what point in the solution process does it become obvious that there is only one solution to this equation?

Step 6 Write an equation to find the time when the ball reaches a height of 200 ft. What happens when you try to solve this impossible situation with the quadratic formula?

It's important to note that a quadratic equation must be in the general form $ax^2 + bx + c = 0$ before you use the quadratic formula.

EXAMPLE B Solve $3x^2 = 5x + 8$.

Solution To use the quadratic formula, first write the equation in the form $ax^2 + bx + c = 0$ and identify the coefficients.

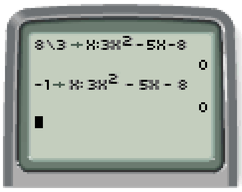
$$3x^2 - 5x - 8 = 0$$
$$a = 3,\ b = -5,\ c = -8$$

Substitute a, b, and c into the quadratic formula.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(3)(-8)}}{2(3)}$$
$$x = \frac{5 \pm \sqrt{121}}{6}$$
$$x = \frac{5 \pm 11}{6}$$
$$x = \frac{5+11}{6} = \frac{8}{3} \text{ or } x = \frac{5-11}{6} = -1$$

The solutions are $x = \frac{8}{3}$ or $x = -1$.

To check your work, substitute these values into the original equation. Here's a way to use your calculator to check.



Remember, you can find exact solutions to some quadratic equations by factoring. However, most don't factor easily. The quadratic formula can be used to solve any quadratic equation.

EXERCISES

Practice Your Skills

1. Solve.

a. $(x - 2.3)^2 = 25$ **b.** $(x + 4.45)^2 = 12.25$ **c.** $\left(x - \frac{3}{4}\right)^2 = \frac{25}{16}$

2. Rewrite each equation in general form, $ax^2 + bx + c = 0$. Identify a, b, and c.

a. $3x^2 - 13x = 10$ **b.** $x^2 - 13 = 5x$ **c.** $3x^2 + 5x = -1$

d. $3x^2 - 2 - 3x = 0$ **e.** $14(x - 4) - (x + 2) = (x + 2)(x - 4)$

3. Evaluate each expression using your calculator. Round your answers to the nearest thousandth.

a. $\frac{-30 + \sqrt{30^2 - 4(5)(3)}}{2(5)}$ **b.** $\frac{-30 - \sqrt{30^2 - 4(5)(3)}}{2(5)}$

c. $\frac{8 - \sqrt{(-8)^2 - 4(1)(-2)}}{2(1)}$ **d.** $\frac{8 + \sqrt{(-8)^2 - 4(1)(-2)}}{2(1)}$

4. Solve by any method.

a. $x^2 - 6x + 5 = 0$ **b.** $x^2 - 7x - 18 = 0$ **c.** $5x^2 + 12x + 7 = 0$

5. Use the roots of the equations in Exercise 4 to write each of these functions in factored form, $y = a(x - r_1)(x - r_2)$.

a. $y = x^2 - 6x + 5$ **b.** $y = x^2 - 7x - 18$ **c.** $y = 5x^2 + 12x + 7$

Reason and Apply

6. Beth uses the quadratic formula to solve an equation and gets

$$x = \frac{-9 \pm \sqrt{9^2 - 4(1)(10)}}{2(1)}$$

a. Write the quadratic equation Beth started with.

b. Write the simplified forms of the exact answers.

c. What are the x-intercepts of the graph of this quadratic function?

7. Write a quadratic function whose graph has these x-intercepts.

a. 3 and –3 **b.** 4 and $-\frac{2}{5}$ **c.** r_1 and r_2

8. Use the quadratic formula to find the zeros of $y = 2x^2 + 2x + 5$. Explain what happens. Graph $y = 2x^2 + 2x + 5$ to confirm your observation. How can you recognize this situation before using the quadratic formula?

9. Write a quadratic function that has no x-intercepts.

10. Show that the mean of the two solutions provided by the quadratic formula is $-\frac{b}{2a}$. Explain what this tells you about a graph.

11. These data give the amount of water in a draining bathtub and the amount of time after the plug was pulled.

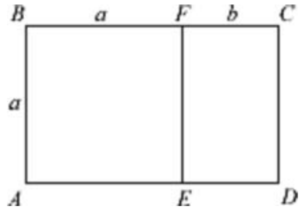
Time (min) x	1	1.5	2	2.5
Amount of water (L) y	38.4	30.0	19.6	7.2

a. Write a function that gives the amount of water as a function of time.

b. How much water was in the tub when the plug was pulled?

c. How long did it take the tub to empty?

12. A **golden rectangle** is a rectangle that can be divided into a square and another rectangle that is also a golden rectangle similar to the original. In the figure at right, *ABCD* is a golden rectangle because it can be divided into square *ABFE* and golden rectangle *FCDE*. Setting up a proportion of the side lengths of the similar rectangles leads to $\frac{a}{a+b} = \frac{b}{a}$. Let $b = 1$ and solve this equation for a.



History CONNECTION

Many people and cultures throughout history have felt that the golden rectangle is one of the most visually pleasing geometric shapes. It was used in the architectural designs of the Cathedral of Notre Dame in Paris, as well as in music and famous works of art. It is believed that the early Egyptians used the ratio when building their pyramids, temples, and tombs, and that they knew the value of the **golden ratio** (the ratio of the length of the golden rectangle to the width) to be $\frac{1+\sqrt{5}}{2}$.

The Fibonacci Fountain in Bowie, Maryland, was designed by mathematician Helaman Ferguson using Fibonacci numbers and the golden ratio. It has 14 water spouts arranged horizontally at intervals proportional to Fibonacci numbers.

Review

13. Complete each equation.

a. $x^2 + \underline{?} + 49 = (x + \underline{?})^2$

b. $x^2 - 10x + \underline{?} = (\underline{?})^2$

c. $x^2 + 3x + \underline{?} = (\underline{?})^2$

d. $2x^2 + \underline{?} + 8 = 2(x^2 + \underline{?} + \underline{?}) = \underline{?}\,(x + \underline{?})^2$

14. Find the inverse of each function. (The inverse does not need to be a function.)

a. $y = (x + 1)^2$ **b.** $y = (x + 1)^2 + 4$ **c.** $y = x^2 + 2x - 5$

15. Convert these quadratic functions to general form.

a. $y = (x - 3)(2x + 5)$

b. $y = -2(x - 1)^2 + 4$

16. A 20 ft ladder leans against a building. Let x represent the distance between the building and the foot of the ladder, and let y represent the height the ladder reaches on the building.

a. Write an equation for y in terms of x.

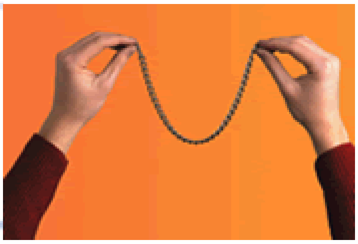
b. Find the height the ladder reaches on the building if the foot of the ladder is 10 ft from the building.

c. Find the distance of the foot of the ladder from the building if the ladder must reach 18 ft up the wall.

17. **APPLICATION** The main cables of a suspension bridge typically hang in the shape of parallel parabolas on both sides of the roadway. The vertical support cables, labeled *a-k*, are equally spaced, and the center of the parabolic cable touches the roadway at *f*. If this bridge has a span of 160 ft between towers, and the towers reach a height of 75 ft above the road, what is the length of each support cable, *a-k*? What is the total length of vertical support cable needed for the portion of the bridge between the two towers?

Engineering CONNECTION

The roadway of a suspension bridge is suspended, or hangs, from large steel support cables. By itself, a cable hangs in the shape of a *catenary* curve. However, with the weight of a roadway attached, the curvature changes, and the cable hangs in a parabolic curve. It is important for engineers to ensure that cables are the correct lengths to make a level roadway.



A chain hangs in the shape of a catenary curve.

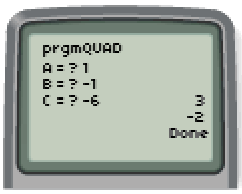
Project

CALCULATOR PROGRAM FOR THE QUADRATIC FORMULA

Write a calculator program that uses the quadratic formula to solve equations. The program should prompt the user to input values for a, b, and c for a quadratic equation in the form $ax^2 + bx + c = 0$, and it should calculate and display the two solutions. Your program may be quite elaborate or very simple.

Your project should include

- A written record of the steps your program uses.
- An explanation of how the program works.
- The results of solving at least two equations by hand and with your program to verify that your program works.



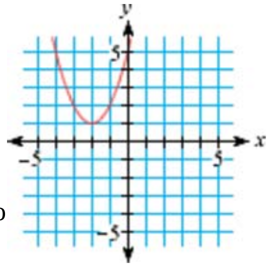
LESSON 7.5

Complex Numbers

Things don't turn up in the world until somebody turns them up.

JAMES A. GARFIELD

You have explored several ways to solve quadratic equations. You can find the x-intercepts on a graph, you can solve by completing the square, or you can use the quadratic formula. What happens if you try to use the quadratic formula on an equation whose graph has no x-intercepts?



The graph of $y = x^2 + 4x + 5$ at right shows that this function has no x-intercepts. Using the quadratic formula to try to find x-intercepts, you get

$$x = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2(1)} = \frac{-4 \pm \sqrt{-4}}{2}$$

How do you take the square root of a negative number? The two numbers $\frac{-4 + \sqrt{-4}}{2}$ and $\frac{-4 - \sqrt{-4}}{2}$ are unlike any of the numbers you have worked with this year- they are nonreal, but they are still numbers. In the development of mathematics, new sets of numbers have been defined in order to solve problems. Mathematicians have defined fractions and not just whole numbers, negative numbers and not just positive numbers, irrational numbers and not just fractions. For the same reasons, we also have square roots of negative numbers, not just square roots of positive numbers. Numbers that include the real numbers as well as the square roots of negative numbers are called **complex numbers.**

History CONNECTION

Since the 1500s, the square root of a negative number has been called an **imaginary number.** In the late 1700s, the Swiss mathematician Leonhard Euler (1707-1783) introduced the symbol i to represent $\sqrt{-1}$. He wrote:

> It is evident that we cannot rank the square root of a negative number amongst possible numbers, and we must therefore say that it is an impossible quantity. . . . But notwithstanding this these numbers present themselves to the mind; they exist in our imagination, and we still have a sufficient idea of them; since we know that by $\sqrt{-4}$ is meant a number which, multiplied by itself, produces –4; for this reason also, nothing prevents us from making use of these imaginary numbers, and employing them in calculation.

Leonhard Euler

Defining imaginary numbers made it possible to solve previously unsolvable problems.

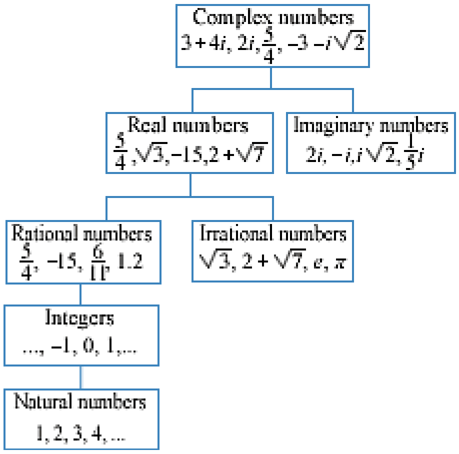
To express the square root of a negative number, we use an **imaginary unit** called i, defined by $i^2 = -1$ or $i = \sqrt{-1}$. You can rewrite $\sqrt{-4}$ as $\sqrt{4} \cdot \sqrt{-1}$, or $2i$. Therefore, you can write the two solutions to the quadratic equation above as the complex numbers $\frac{-4 + 2i}{2}$ and $\frac{-4 - 2i}{2}$, or $-2 + i$ and $-2 - i$. These two solutions are a **conjugate pair.** That is, one is $a + bi$ and the other is $a - bi$. The two numbers in a complex pair are **complex conjugates.** Why will nonreal solutions to the quadratic formula always give answers that are a conjugate pair?

Roots of polynomial equations can be real numbers or nonreal complex numbers, or there may be some of each. If the polynomial has real coefficients, any nonreal roots will come in conjugate pairs such as $2i$ and $-2i$ or $3 + 4i$ and $3 - 4i$.

Complex Numbers

A **complex number** is a number in the form $a + bi$, where a and b are real numbers and $i = \sqrt{-1}$.

For any complex number in the form $a + bi$, a is the real part and b is the imaginary part. The set of complex numbers contains all real numbers and all imaginary numbers. This diagram shows the relationship between these numbers and some other sets you may be familiar with, as well as examples of numbers within each set.



EXAMPLE Solve $x^2 + 3 = 0$.

Solution

You can use the quadratic formula, or you can isolate x^2 and take the square root of both sides.

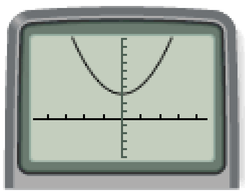
$$\begin{aligned} x^2 + 3 &= 0 \\ x^2 &= -3 \\ x &= \pm\sqrt{-3} \\ x &= \pm\sqrt{3} \cdot \sqrt{-1} \\ x &= \pm\sqrt{3} \cdot i \\ x &= \pm i\sqrt{3} \end{aligned}$$

To check the two solutions, substitute them into the original equation.

$$\begin{aligned} x^2 + 3 &= 0 \\ (i\sqrt{3})^2 + 3 &\stackrel{?}{=} 0 \\ i^2 \cdot 3 + 3 &\stackrel{?}{=} 0 \\ -1 \cdot 3 + 3 &\stackrel{?}{=} 0 \\ -3 + 3 &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} x^2 + 3 &= 0 \\ (-i\sqrt{3})^2 + 3 &\stackrel{?}{=} 0 \\ i^2 \cdot 3 + 3 &\stackrel{?}{=} 0 \\ -1 \cdot 3 + 3 &\stackrel{?}{=} 0 \\ -3 + 3 &\stackrel{?}{=} 0 \\ 0 &= 0 \end{aligned}$$

The two imaginary numbers $\pm i\sqrt{3}$ are solutions to the original equation, but because they are not real, the graph of $y = x^2 + 3$ shows no x-intercepts.



[–4.7, 4.7, 1, –5, 10, 1]

Complex numbers are used to model many applications, particularly in science and engineering. To measure the strength of an electromagnetic field, a real number represents the amount of electricity, and an imaginary number represents the amount of magnetism. The state of a component in an electronic circuit is also measured by a complex number, where the voltage is a real number and the current is an imaginary number. The properties of calculations with complex numbers apply to these types of physical states more accurately than calculations with real numbers do. In the investigation you'll explore patterns in arithmetic with complex numbers.

The Heart Revealed: Portrait of Tita Thirifays (1936), by Belgian Surrealist painter Rene Magritte (1898-1967), is a portrait with an imaginary element.

Investigation
Complex Arithmetic

When computing with complex numbers, there are conventional rules similar to those you use when working with real numbers. In this investigation you will discover these rules. You may use your calculator to check your work or to explore other examples. [See Calculator Note 7E to learn how to enter complex numbers into your calculator.]

Part 1: Addition and Subtraction

Addition and subtraction of complex numbers is similar to combining like terms. Use your calculator to add these complex numbers. Make a conjecture about how to add complex numbers without a calculator.

a. $(2 - 4i) + (3 + 5i)$ **b.** $(7 + 2i) + (-2 + i)$

c. $(2 - 4i) - (3 + 5i)$ **d.** $(4 - 4i) - (1 - 3i)$

Part 2: Multiplication

Use your knowledge of multiplying binomials to multiply these complex numbers. Express your products in the form $a + bi$. Recall that $i^2 = -1$.

a. $(2 - 4i)(3 + 5i)$ **b.** $(7 + 2i)(-2 + i)$

c. $(2 - 4i)^2$ **d.** $(4 - 4i)(1 - 3i)$

Part 3: The Complex Conjugates

Recall that every complex number $a + bi$ has a complex conjugate, $a - bi$. Complex conjugates have some special properties and uses.

Each expression below shows either the sum or product of a complex number and its conjugate. Simplify these expressions into the form $a + bi$, and generalize what happens.

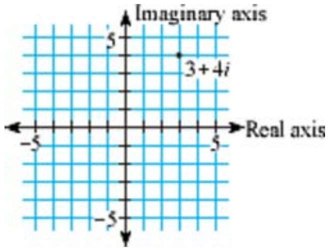
a. $(2 - 4i) + (2 + 4i)$ **b.** $(7 + 2i)(7 + 2i)$
c. $(2 - 4i)(2 + 4i)$ **d.** $(-4 + 4i)(-4 - 4i)$

Part 4: Division

To divide two complex numbers in the form $a + bi$, you need to eliminate the imaginary part of the denominator. First, use your work from Part 3 to decide how to change each denominator into a real number. Once you have a real number in the denominator, divide to get an answer in the form $a + bi$.

a. $\frac{7+2i}{1-i}$ **b.** $\frac{2-5i}{3+4i}$ **c.** $\frac{2-i}{8-6i}$ **d.** $\frac{2-4i}{2+4i}$

You cannot graph a complex number, such as $3 + 4i$, on a real number line, but you can graph it on a **complex plane,** where the horizontal axis is the **real axis** and the vertical axis is the **imaginary axis.** In the graph at right, $3 + 4i$ is located at the point with coordinates (3, 4). Any complex number $a + bi$ has (a, b) as its coordinates on a complex plane.
You'll observe some properties of a complex plane in the exercises.



EXERCISES

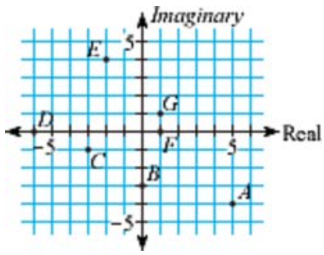
Practice Your Skills

1. Add or subtract.

a. $(5 - 1i) + (3 + 5i)$ **b.** $(6 + 2i) - (-1 + 2i)$
c. $(2 + 3i) + (2 - 5i)$ **d.** $(2.35 + 2.71i) - (4.91 + 3.32i)$

2. Multiply.

a. $(5 - 1i)(3 + 5i)$ **b.** $6\ (-1 + 2i)$
c. $3i\ (2 - 5i)$ **d.** $(2.35 + 2.71i)(4.91 + 3.32i)$



3. Find the conjugate of each complex number.

a. $5 - i$ **b.** $-1 + 2i$
c. $2 + 3i$ **d.** $-2.35 - 2.71i$

4. Name the complex number associated with each point, *A* through *G*, on the complex plane at right.

5. Draw **Venn diagrams** to show the relationships between these sets of numbers.

a. real numbers and complex numbers

b. rational numbers and irrational numbers

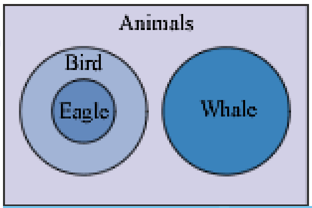
c. imaginary numbers and complex numbers

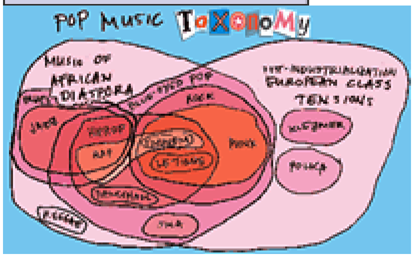
d. imaginary numbers and real numbers

e. complex numbers, real numbers, and imaginary numbers

History CONNECTION

In his book *Symbolic Logic*, English logician John Venn (1834-1923) proposed using diagrams as a method of representing logic relationships. For example, this diagram shows "if it's an eagle, then it's a bird," and "if it's a bird, then it's not a whale." It must also be true, therefore, that "if it's an eagle, then it's not a whale." A Venn diagram of this situation shows this conclusion clearly. Venn diagrams have become a tool for representing many kinds of relationships.





This Venn diagram by Jason Luz shows a taxonomy of music. Do you think speed metal should be included? The Beatles? Madonna? Make your own Venn diagram.

Reason and Apply

6. Rewrite this quadratic equation in general form.

$$(x-(2+i))(x-(2-i))=0$$

7. Use the definitions $i=\sqrt{-1}$ and $i^2=-1$ to rewrite each power of i as 1, i, –1, or $-i$.

a. i^3 **b.** i^4

c. i^5 **d.** i^{10}

8. ***Mini-Investigation*** Plot the numbers i, i^2, i^3, i^4, i^5, i^6, i^7, and i^8 on a complex plane. What pattern do you see? What do you expect the value of i^{17} to be? Verify your conjecture.

9. Rewrite the quotient $\frac{2+3i}{2-i}$ in the form $a+bi$.

10. Solve each equation. Label each solution as real, imaginary, and/or complex.

a. $x^2-4x+6=0$ **b.** $x^2+1=0$

c. $x^2+x=-1$ **d.** $x^2-1=0$

11. Write a quadratic function in general form that has the given zeros, and leading coefficient of 1.

a. $x=-3$ and $x=5$ **b.** $x=-3.5$ (double root)

c. $x=5i$ and $x=-5i$ **d.** $x=2+i$ and $x=2-i$

12. Write a quadratic function in general form that has a zero of $x=4+3i$ and whose graph has y-intercept 50.

13. Solve.

a. $x^2-10ix-9i^2=0$ **b.** $x^2-3ix=2$

c. Why don't the solutions to 13a and 13b come in conjugate pairs?

14. The quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, provides solutions to $ax^2 + bx + c = 0$. Make up some rules involving a, b, and c that determine each of these conditions.

a. The solutions are nonreal.

b. The solutions are real.

c. There is only one real solution.

15. Use these recursive formulas to find the first six terms (z_0 to z_5) of each sequence. Describe what happens in the long run for each sequence.

a. $z_0 = 0$
$z_n = z_{n-1}^2 + 0$ where $n \geq 1$

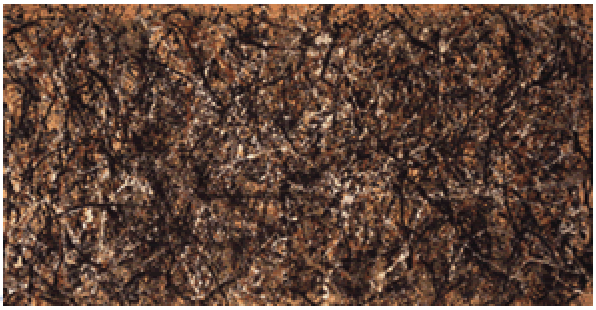
b. $z_0 = 0$
$z_n = z_{n-1}^2 + i$ where $n \geq 1$

c. $z_0 = 0$
$z_n = z_{n-1}^2 + 1 - i$ where $n \geq 1$

d. $z_0 = 0$
$z_n = z_{n-1}^2 + 0.2 + 0.2i$ where $n \geq 1$

Mathematics CONNECTION

Fractals can appear quite complicated, yet they are generated by simple rules. Complex numbers c which converge according to the recursive formula $z_0 = 0$ and $z_n = z_{n-1}^2 + c$, where $n \geq 1$, are members of the Mandelbrot set. In the picture picture of the Mandelbrot set on the opposite page, these convergent points are plotted as black points in a complex plane. Polish mathematician Benoit Mandelbrot (b 1924) noted that fractals aren't just a mathematical curiosity but, rather, the geometry of nature. Clouds, coastlines, and trees can be described using fractal geometry. Fractals are used in medicine to study the growth of cancer tissue, in art to date early paintings, and in computer programming to compress large sets of data.



American abstract painter Jackson Pollock (1912-1956) created pieces such as *Number 31* (1950) using oil and enamel paints on unprimed canvas. Physicist and abstract artist Richard P. Taylor recently discovered, with the aid of a computer, that Pollock's paintings display the fractal characteristic of self-similarity.

Review

16. Consider the function $y = 2x^2 + 6x - 3$.

a. List the zeros in exact radical form and as approximations to the nearest hundredth.

b. Graph the function and label the exact coordinates of the vertex and points where the graph crosses the x-axis and the y-axis.

17. Consider two positive integers that meet these conditions:

- three times the first added to four times the second is less than 30
- twice the first is less than five more than the second

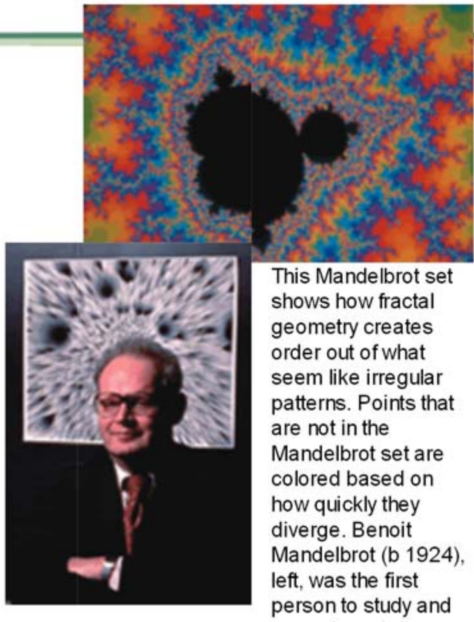
a. Define variables and write a system of linear inequalities that represents this situation.

b. Graph the feasible region.

c. List all integer pairs that satisfy the conditions listed above.

Project

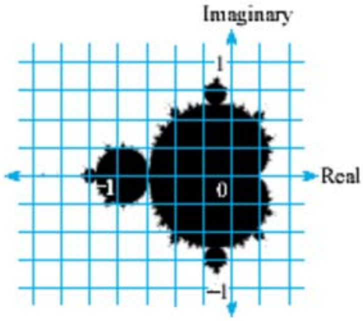
THE MANDELBROT SET



This Mandelbrot set shows how fractal geometry creates order out of what seem like irregular patterns. Points that are not in the Mandelbrot set are colored based on how quickly they diverge. Benoit Mandelbrot (b 1924), left, was the first person to study and name fractal geometry.

You have seen geometric **fractals** such as the Sierpiński triangle, and you may have seen other fractals that look much more complicated. The Mandelbrot set is a famous fractal that relies on repeated calculations with complex numbers. To create the Mandelbrot set, you use the recursive formula $z_0 = 0$ and $z_n = z_{n-1}{}^2 + c$ where $n \geq 1$. Depending on the complex number you choose for the constant, c, one of two things will happen: either the magnitude of the values of z will get increasingly large, or they will not. (The magnitude of a complex number is defined as its distance from the origin of the complex plane, or $\sqrt{a^2 + b^2}$.) You already explored a few values of c in Exercise 15. Try a few more. Which values of c make the magnitude of z get increasingly large? Which values of c make z converge to a single value or alternate between values?

Use your calculator to determine what happens if $z_0 = 0$ and $c = 0.25$. What happens if c or z is a complex number, for example, if $z_0 = 0$ and $c = -0.4 + 0.5i$?



The Mandelbrot set is all of the values of c that do not make the magnitude of z get increasingly large. If you plot these points on a complex plane, then you'll get a pattern that looks like this one. Your project is to choose a small region on the boundary of the black area of this graph and create a graph of that smaller region. [▸ **Calculator Note 7F** includes a program that analyzes every point in the window to determine whether it is in the Mandelbrot set. ◂] Look at this graph, select a window, and then run the program. It may take several hours.

Your project should include

- A sketch of your graph.
- A report that describes any similarities between your portion of the Mandelbrot set and the complete graph shown above.
- Any additional research you do on the Mandelbrot set, or fractals in general.

You can learn more about the Mandelbrot set and other fractals by using the links at www.keymath.com/DAA.

LESSON

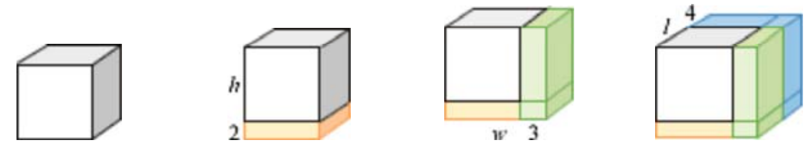
7.6

Factoring Polynomials

Ideas are the factors that lift civilization.

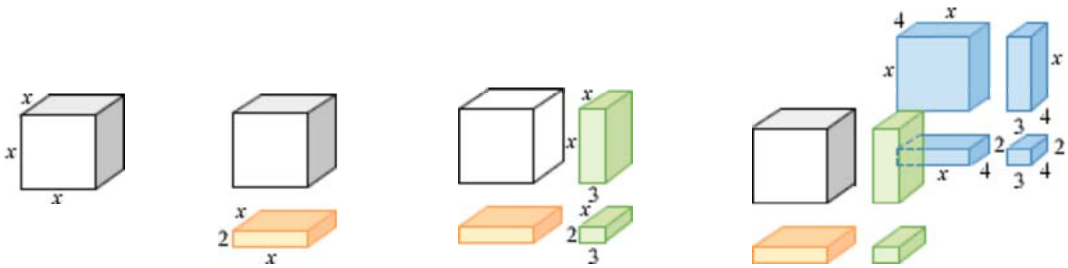
JOHN H. VINCENT

Imagine a cube with any side length. Imagine increasing the height by 2 cm, the width by 3 cm, and the length by 4 cm.



Imagine a cube. Increase its height 2 cm. Increase the width 3 cm. Increase the length 4 cm.

The starting figure is a cube, so you can let x be the length of each its sides. So, $l = w = h = x$. The volume of the starting figure is x^3. To find the volume of the expanded box, you can see it as the sum of the volumes of eight different boxes. You find the volume of each piece by multiplying length by width by height.

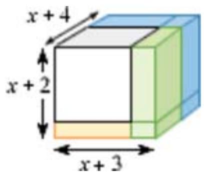


The total expanded volume is this sum:

$$V = x^3 + 2x^2 + 3x^2 + 6x + 4x^2 + 8x + 12x + 24 = x^3 + 9x^2 + 26x + 24$$

You can also think of the expanded volume as the product of the new height, width, and length.

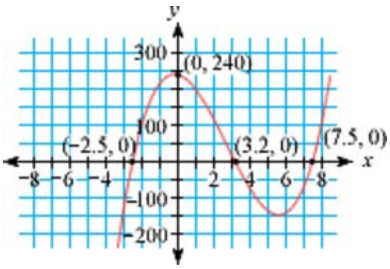
$$V = (x + 2)(x + 3)(x + 4)$$

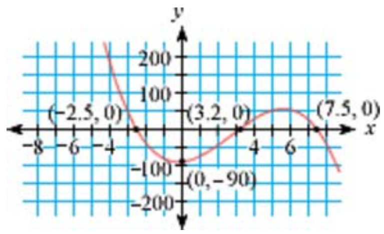


This function in factored form is equivalent to the polynomial function in general form. (Try graphing both functions on your calculator.)

You already know that there is a relationship between the factored form of a quadratic equation, and the roots and x-intercepts of that quadratic equation. In this lesson you will learn how to write higher-degree polynomial equations in factored form when you know the roots of the equation. You'll also discover useful techniques for converting a polynomial in general form to factored form.

A 3rd-degree polynomial function is called a **cubic function.** Let's examine the features of the graph of a cubic function. The graphs of the two cubic functions below have the same *x*-intercepts: –2.5, 3.2, and 7.5. So both functions have the factored form $y = a(x + 2.5)(x - 7.5)(x - 3.2)$, but the vertical scale factor, *a,* is different for each function.





As you know by now, one way to find *a* is to substitute coordinates of one other point, such as the *y*-intercept, into the function.

The curve on the left has *y*-intercept (0, 240). Substituting this point into the equation gives $240 = a(2.5)(-7.5)(-3.2)$. Solving for *a*, you get $a = 4$. So the equation of the cubic function on the left is

$$y = 4(x + 2.5)(x - 7.5)(x - 3.2)$$

The curve on the right has *y*-intercept (0, –90). Substituting this point into the equation gives $-90 = a(2.5)(-7.5)(-3.2)$. So $a = -1.5$, and the equation of the cubic function on the right is

$$y = -1.5(x + 2.5)(x - 7.5)(x - 3.2)$$

The factored form of a polynomial function tells you the zeros of the function and the *x*-intercepts of the graph of the function. Recall that zeros are solutions to the equation $f(x) = 0$. Factoring, if a polynomial can be factored, is one strategy for finding the real solutions of a polynomial equation. You will practice writing a higher-degree polynomial function in factored form in the investigation.

English sculptor Cornelia Parker (b 1956) creates art from damaged objects that have cultural or historical meaning. *Mass* (*Colder Darker Matter*) (1997) is made of the charred remains of a building struck by lightning-the building has been reduced to its charcoal factors. The sculpture looks flat when viewed from the front, but it is actually constructed in the shape of a cube.

Cornelia Parker *Mass* (*Colder Darker Matter*) (1997) charcoal, wire, and black string, Collection of Phoenix Art Museum, Gift of Jan and Howard Hendler 2002.1.

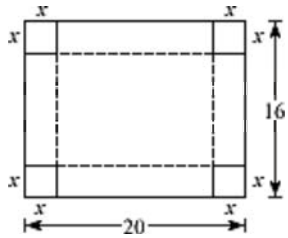
Investigation

The Box Factory

You will need

- graph paper
- scissors

What are the different ways to construct an open-top box from a 16-by-20-unit sheet of material? What is the maximum volume this box can have? What is the minimum volume? Your group will investigate this problem by constructing open-top boxes using several possible integer values for x.



Procedure Note

1. Cut several 16-by-20–unit rectangles out of graph paper.
2. Choose several different values for x.
3. For each value of x, construct a box by cutting squares with side length x from each corner and folding up the sides.

Step 1 Follow the procedure note to construct several different-size boxes from 16-by-20-unit sheets of paper. Record the dimensions of each box and calculate its volume. Make a table to record the x-values and volumes of the boxes.

Step 2 For each box, what are the length, width, and height, in terms of x? Use these expressions to write a function that gives the volume of a box as a function of x.

Step 3 Graph your volume function from Step 2. Plot your data points on the same graph. How do the points relate to the function?

Step 4 What is the degree of this function? Give some reasons to support your answer.

Step 5 Locate the x-intercepts of your graph. (There should be three.) Call these three values r_1, r_2, and r_3. Use these values to write the function in the form $y = (x - r_1)(x - r_2)(x - r_3)$.

Step 6 Graph the function from Step 5 with your function from Step 2. What are the similarities and differences between the graphs? How can you alter the function from Step 5 to make both functions equivalent?

Step 7 What happens if you try to make boxes by using the values r_1, r_2, and r_3 as x? What domain of x-values makes sense in this context? What x-value maximizes the volume of the box?

The connection between the roots of a polynomial equation and the x-intercepts of a polynomial function helps you factor any polynomial that has real roots.

EXAMPLE

Find the factored form of each function.

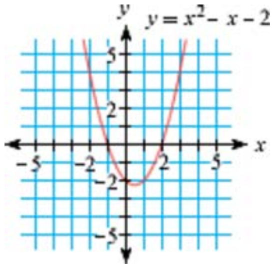
a. $y = x^2 - x - 2$

b. $y = 4x^3 + 8x^2 - 36x - 72$

▸ Solution

You can find the x-intercepts of each function by graphing. The x-intercepts tell you the real roots, which help you factor the function.

a. The graph shows that the x-intercepts are –1 and 2. Because the coefficient of the highest-degree term, x^2, is 1, the vertical scale factor is 1. The factored form is $y = (x + 1)(x - 2)$.



You can verify that the expressions $x^2 - x - 2$ and $(x + 1)(x - 2)$ are equivalent by graphing $y = x^2 - x - 2$ and $y = (x + 1)(x - 2)$. You can also check your work algebraically by finding the product $(x + 1)(x - 2)$. This rectangle diagram confirms that the product is $x^2 - x - 2$.

	x	1
x	x^2	$1x$
-2	$-2x$	-2

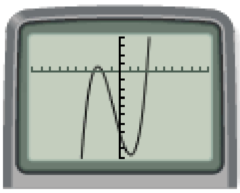
b. The x-intercepts are –3, –2, and 3. So, you can write the function as

$$y = a(x + 3)(x + 2)(x - 3)$$

Because the leading coefficient needs to be 4, the vertical scale factor is also 4.

$$y = 4(x + 3)(x + 2)(x - 3)$$

To check your answer, you can compare graphs or algebraically find the product of the factors.



[–10, 10, 1, –100, 40, 10]

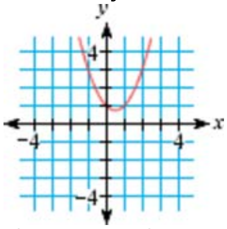
In the example, you converted a function from general form to factored form by using a graph and looking for the x-intercepts. This method works especially well when the zeros are integer values. Once you know the zeros of a polynomial function, r_1, r_2, r_3, and so on, you can write the factored form,

$$y = a(x - r_1)(x - r_2)(x - r_3)\ldots$$

You can also write a polynomial function in factored form when the zeros are not integers, or even when they are nonreal.

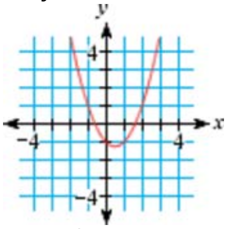
Polynomials with real coefficients can be separated into three types: polynomials that can't be factored with real numbers; polynomials that can be factored with real numbers, but the roots are not "nice" integer or rational values; and polynomials that can be factored and have integer or rational roots. For example, consider these cases of quadratic functions:

$y = x^2 - x + 1$



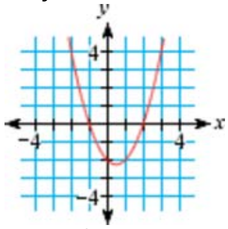
If the graph of a quadratic function does not intercept the x-axis, then you cannot factor the polynomial using real numbers. However, you can use the quadratic formula to find the complex zeros, which will be a conjugate pair.

$y = x^2 - x - 1$



If the graph of a quadratic function intercepts the x-axis, but not at integer or rational values, then you can use the quadratic formula to find the real zeros.

$y = x^2 - x - 2$



If the graph of a quadratic function intercepts the x-axis at integer or rational values, then you can use the x-intercepts to factor the polynomial. This is often quicker and easier than using the quadratic formula or a rectangle diagram.

What happens when the graph of a quadratic function has exactly one point of intersection with the x-axis?

EXERCISES

Practice Your Skills

1. Without graphing, find the x-intercepts and the y-intercept for the graph of each equation. Check each answer by graphing.

a. $y = -0.25(x + 1.5)(x + 6)$

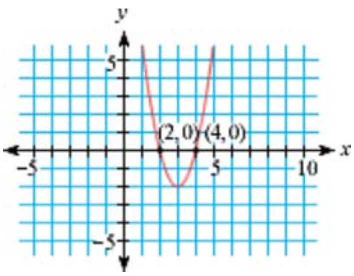
b. $y = 3(x - 4)(x - 4)$

c. $y = -2(x - 3)(x + 2)(x + 5)$

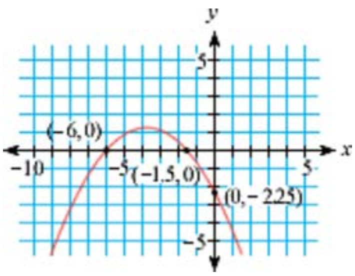
d. $y = 5(x + 3)(x + 3)(x - 3)$

2. Write the factored form of the quadratic function for each graph. Don't forget the vertical scale factor.

a.



b.



3. Convert each polynomial function to general form.

a. $y = (x - 4)(x - 6)$

b. $y = (x - 3)(x - 3)$

c. $y = x(x + 8)(x - 8)$

d. $y = 3(x + 2)(x - 2)(x + 5)$

4. Given the function $y = 2.5(x - 7.5)(x + 2.5)(x - 3.2)$,

a. Find the x-intercepts without graphing.

b. Find the y-intercept without graphing.

c. Write the function in general form.

d. Graph both the factored form and the general form of the function to check your work.

Reason and Apply

5. Use your work from the investigation to answer these questions.

a. What x-value maximizes the volume for your box? What is the maximum volume possible?

b. What x-value or values give a volume of 300 cubic units?

c. The portion of the graph with domain $x > 10$ shows positive volume. What does this mean in the context of the problem?

d. Explain the meaning of the parts of the graph showing negative volume.

These boxes, on display during the 2002 Cultural Olympiad in Salt Lake City, Utah, through the organization "Children Beyond Borders," were created by children with disabilities in countries worldwide. The children decorated identical 4-inch square boxes with their own creative visions, often in scenes from their countries and themes of love and unity.

1. The original cardboard box; **2.** © Diana Edna Cruz Yunes, *Corazon Mío, Mi Corazon*/Mexico; **3.** © Manal Deibes, *Are You Hungry?*/Jordan; **4.** © Leslie Hendricks, *Art Is What Makes the World Go 'Round*/USA; **5.** © Earl Hasith Vanabona, *Untitled/Sri Lanka;* **6.** © Renato Pinho, Ocean/Portugal. Participating artists of VSA arts *(www.vsarts.org)*

6. Write each polynomial as a product of factors.

a. $4x^2 - 88x + 480$

b. $6x^2 - 7x - 5$

c. $x^3 + 5x^2 - 4x - 20$

d. $2x^3 + 16x^2 + 38x + 24$

e. $a^2 + 2ab + b^2$

f. $x^2 - 64$

g. $x^2 + 64$

h. $x^2 - 7$

i. $x^2 - 3x$

7. Sketch a graph for each situation if possible.

a. a quadratic function with only one real zero

b. a quadratic function with no real zeros

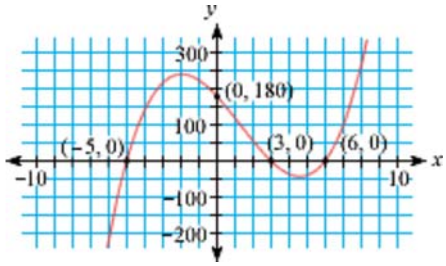
c. a quadratic function with three real zeros

d. a cubic function with only one real zero

e. a cubic function with two real zeros

f. a cubic function with no real zeros

8. Consider the function in this graph.



a. Write the equation of a polynomial function that has the x-intercepts shown in the graph. Use a for the vertical scale factor.

b. Use the y-intercept to determine the vertical scale factor. Write the function from 8a, replacing a with the value of the vertical scale factor.

c. Imagine that this graph is translated up 100 units. Write the equation of the image.

d. Imagine that this graph is translated left 4 units. Write the equation of the image.

9. **APPLICATION** The way you taste certain flavors is a genetic trait inherited from your parents. For instance, the ability to taste the bitter compound phenyltheocarbamide (PTC) is inherited as a dominant trait in humans. In the United States, approximately 70% of the population can taste PTC, whereas 30% cannot.

	T	t
T	?	?
t	?	?

Every person inherits a pair of genes from their parents. Let T represent the gene for tasters (dominant), and let t represent the gene for nontasters (recessive). The presence of at least one T means that a person can taste PTC.

a. Complete the rectangle diagram of possible gene-pair combinations. Using the variables T and t, what algebraic expression does this diagram represent?

b. The sum of all possible gene-pair combinations must equal 1, or 100%, for the entire population. Write an equation to express this relationship.

c. Use the fact that 70% of the U.S. population are tasters to write an equation that you can use to solve for t.

d. Solve for t, the frequency of the recessive gene in the population.

e. What is the frequency of the dominant gene in the population?

f. What percentage of the U.S. population has the gene-pair TT?

Review

10. Is it possible to find a quadratic function that contains the points (–4, –2), (–1, 7), and (2, 16)? Explain why or why not.

11. Find the quadratic function whose graph has vertex (–2, 3) and contains the point (4, 12).

12. Find all real solutions.

a. $x^2 = 50.4$ **b.** $x^4 = 169$ **c.** $(x - 2.4)^2 = 40.2$ **d.** $x^3 = -64$

13. Algebraically find the inverse of each function. Then choose a value of x and check your answer.

a. $f(x) = \frac{2}{3}(x + 5)$ **b.** $g(x) = -6 + (x + 3)^{2/3}$ **c.** $h(x) = 7 - 2^x$

14. Use the finite differences method to find the function that generates this table of values. Explain your reasoning.

x	2.2	2.6	3.0	3.4
$f(x)$	– 4.5	– 5.5	– 6.5	- 7.5

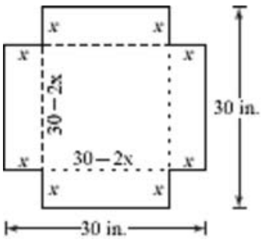
LESSON 7.7

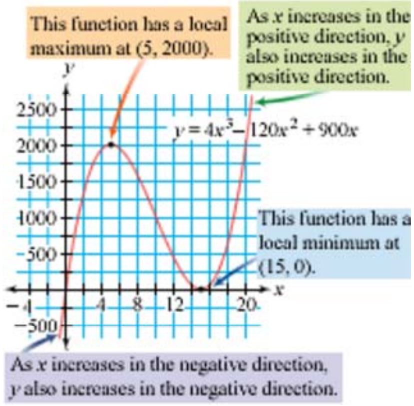
Higher-Degree Polynomials

It is good to have an end to journey towards, but it is the journey that matters in the end.

URSULA K. LEGUIN

Polynomials with degree 3 or higher are called higher-degree polynomials. Frequently, 3rd-degree polynomials are associated with volume measures, as you saw in Lesson 7.6. If you create a box by removing small squares of side length x from each corner of a square piece of cardboard that is 30 inches on each side, the volume of the box in cubic inches is modeled by the function $y = x(30 - 2x)^2$, or $y = 4x^3 - 120x^2 + 900x$. The zero-product property tells you that the zeros are $x = 0$ or $x = 15$, the two values of x for which the volume is 0. The x-intercepts on the graph below confirm this.



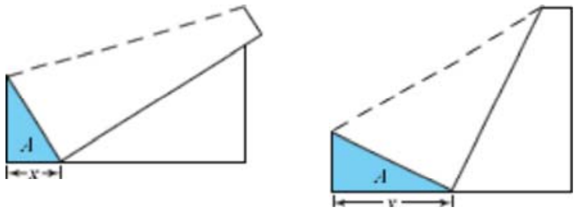


The shape of this graph is typical of the higher-degree polynomial graphs you will work with in this lesson. Note that it has one **local maximum** at (5, 2000) and one **local minimum** at (15, 0). These are the points that are higher or lower than all other points near them. You can also describe the **end behavior**—what happens to $f(x)$ as x takes on larger positive and negative values of x. In the case of this cubic function, as x increases in the positive direction, y also increases in the positive direction. As x increases in the negative direction, y also increases in the negative direction.

Polynomials with real coefficients have graphs that have a y-intercept, and possibly one or more x-intercepts. You can also describe other features of polynomial graphs, such as local maximums or minimums, and end behavior. Maximums and minimums, collectively, are called **extreme values.**

Egyptian-American artist Karim Rashid (b 1960) is a contemporary designer of fine art, as well as commercial products—from trash cans to sofas. This piece, *Softscape* (2001), is the artist's vision of a futuristic living room with chairs, tables, and a television melding together.

Investigation
The Largest Triangle



Take a sheet of notebook paper and orient it such that the longest edge is closest to you. Fold the upper left corner so that it touches some point on the bottom edge. Find the area, A, of the triangle formed in the lower left corner of the paper. What distance, x, along the bottom of the paper produces the triangle with the greatest area?

Work with your group to find a solution. You may want to use strategies you've learned in several lessons in this chapter. Write a report that explains your solution and your group's strategy for finding the largest triangle. Include any diagrams, tables, or graphs that you used.

In the remainder of this lesson you will explore the connections between a polynomial equation and its graph, which will allow you to predict when certain features will occur in the graph.

EXAMPLE A

Find a polynomial function whose graph has x-intercepts 3, 5, and –4, and y-intercept 180. Describe the features of its graph.

▸ Solution

A polynomial function with three x-intercepts has too many x-intercepts to be a quadratic function. It could be a 3rd-, 4th-, 5th-, or higher-degree polynomial function. Consider a 3rd-degree polynomial function, because that is the lowest degree that has three x-intercepts. Use the x-intercepts to write the equation $y = a(x-3)(x-5)(x+4)$ where $a \neq 0$.

Substitute the coordinates of the y-intercept, (0, 180), into this function to find the vertical scale factor.

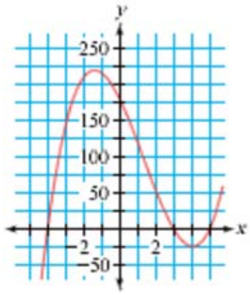
$$180 = a(0-3)(0-5)(0+4)$$
$$180 = a(60)$$
$$a = 3$$

The polynomial function of the lowest degree through the given intercepts is

$$y = 3(x-3)(x-5)(x+4)$$

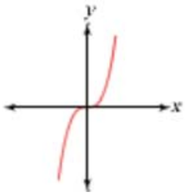
Graph this function to confirm your answer and look for features.

This graph shows a local minimum at about (4, – 25) because that is the lowest point in its immediate neighborhood of x-values. There is also a local maximum at about (–1.5, 220) because that is the highest point in its immediate neighborhood of x-values. The small domain shown in the graph already suggests the end behavior. As x increases in the positive direction, y also increases in the positive direction. As x increases in the negative direction, y also increases in the negative direction. If you increase the domain of this graph to include more x-values at the right and left extremes of the x-axis, you'll see that the graph does continue this end behavior.

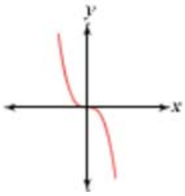


You can identify the degree of many polynomial functions by looking at the shapes of their graphs. Every 3rd-degree polynomial function has essentially one of the shapes shown below. Graph A shows the graph of $y = x^3$. It can be translated, stretched, or reflected. Graph B shows one possible transformation of Graph A. Graphs C and D show the graphs of general cubic functions in the form $y = ax^3 + bx^2 + cx + d$. In Graph C, a is positive, and in Graph D, a is negative.

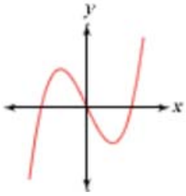
Graph A



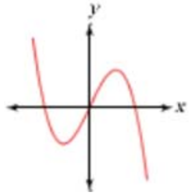
Graph B



Graph C



Graph D



You'll explore the general shapes and characteristics of other higher-degree polynomials in the exercises.

EXAMPLE B

Write a polynomial function with real coefficients and zeros $x = 2$, $x = -5$, and $x = 3 + 4i$.

►*Solution*

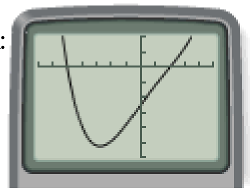
For a polynomial function with real coefficients, complex zeros occur in conjugate pairs, so $x = 3 - 4i$ must also be a zero. In factored form the polynomial function of the lowest degree is

$$y = (x - 2)(x + 5)(x - (3 + 4i))\ (x - (3 - 4i))$$

Multiplying the last two factors to eliminate complex numbers gives $y = (x - 2)\ (x + 5)\ (x^2 - 6x + 25)$. Multiplying all factors gives the polynomial function in general form:

$$y = x^4 - 3x^3 - 3x^2 + 135x - 250$$

Graph this function to check your solution. You can't see the complex zeros, but you can see x-intercepts 2 and –5.



[–7, 5, 1, – 600, 200, 100]

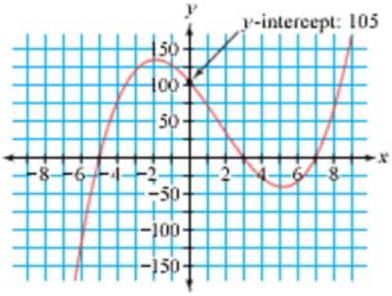
Note that in Example B, the solution was a 4th-degree polynomial function. It had four complex zeros, but the graph had only two x-intercepts, corresponding to the two real zeros. Any polynomial function of degree n always has n complex zeros (including repeated zeros) and at most n x-intercepts. Remember that complex zeros of polynomial functions with real coefficients always come in conjugate pairs.

EXERCISES

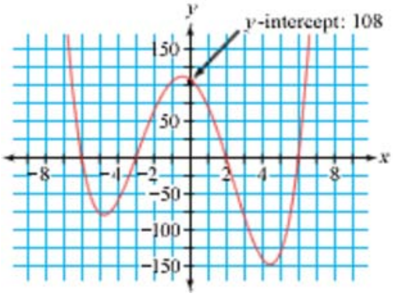
Practice Your Skills

For Exercises 1–4, use these four graphs.

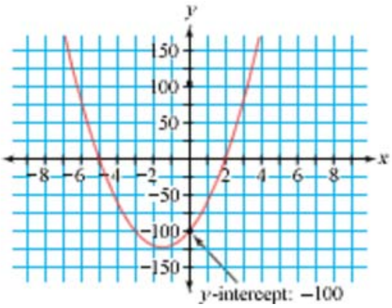
a.



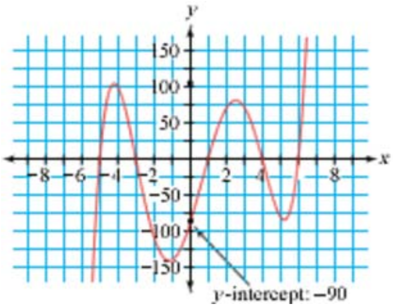
b.



c.



d.



1. Identify the zeros of each function.

2. Give the coordinates of the y-intercept of each graph.

3. Identify the lowest possible degree of each polynomial function.

4. Write the factored form for each polynomial function. Check your work by graphing on your calculator.

American painter Inka Essenhigh (b 1969) calls her works “cyborg mutations.” She draws and paints images, and then sands them and layers them with enamel-based oil paint. This piece, *Green Wave* (2002), contains polynomial-like waves.

Reason and Apply

5. Write polynomial functions with these features.

a. A linear function whose graph has x-intercept 4.

b. A quadratic function whose graph has only one x-intercept, 4.

c. A cubic function whose graph has only one x-intercept, 4.

Andrea Champlin's paintings are often called "cyber" or "digital" landscapes. Can you identify curves that look like polynomials in this painting?

Wandee Love (2001), Andrea Champlin, oil on canvas, 70 in. × 46 in. Courtesy of the artist and Clifford-Smith Gallery, Boston; photo courtesy of the artist.

6. The graph of $y = 2(x - 3)(x - 5)(x + 4)^2$ has x-intercepts 3, 5, and -4 because they are the only possible x-values that make $y = 0$. This is a 4th-degree polynomial, but it has only three x-intercepts. The root $x = -4$ is called a **double root** because the factor $(x + 4)$ occurs twice. Make a complete graph—one that displays all of the relevant features, including local extreme values—of each of the functions in parts a-f.

a. $y = 2(x - 3)(x - 5)(x + 4)^2$

b. $y = 2(x - 3)^2(x - 5)(x + 4)$

c. $y = 2(x - 3)(x - 5)^2(x + 4)$

d. $y = 2(x - 3)^2(x - 5)(x + 4)^2$

e. $y = 2(x - 3)(x - 5)(x + 4)^3$

f. $y = 2(x - 3)(x - 5)^2(x + 4)^3$

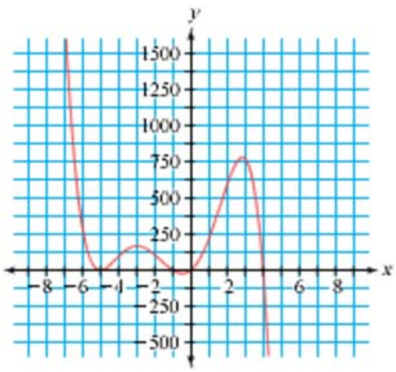
g. Based on your graphs from 6a–f, describe a connection between the power of a factor and what happens at that x-intercept.

7. The graph at right is a complete graph of a polynomial function.

a. How many x-intercepts are there?

b. What is the lowest possible degree of this polynomial function?

c. Write the factored form of this function if the graph includes the points (0, 0), (– 5, 0), (4, 0), (– 1, 0), and (1, 216).



8. Write the lowest-degree polynomial function that has the given set of zeros and whose graph has the given y-intercept.

a. zeros: $x = -4$, $x = 5$, $x = -2$ (double root); y-intercept: -80

b. zeros: $x = -4$, $x = 5$, $x = -2$ (double root); y-intercept: 160

c. zeros: $x = \frac{1}{3}$, $x = -\frac{2}{5}$, $x = 0$; y-intercept: 0

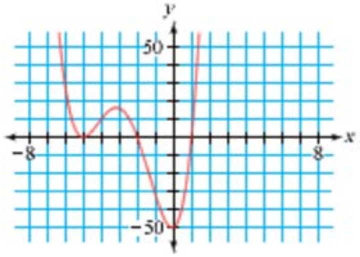
b. zeros: $x = -5i$, $x = -1$ (triple root), $x = 4$; y-intercept: -100

9. Look back at Exercises 1–4. Find the products of the zeros in Exercise 1. How does the value of the leading coefficient, a, relate to the y-intercept, the product of the zeros, and the degree of the function?

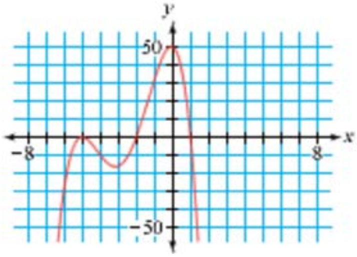
10. A 4th-degree polynomial function has the general form $y = ax^4 + bx^3 + cx^2 + dx + e$ for real values of a, b, c, d, and e, where $a \neq 0$. Graph several 4th-degree polynomial functions by trying different values for each coefficient. Be sure to include positive, negative, and zero values. Make a sketch of each different type of curve you get. Concentrate on the shape of the curve. You do not need to include axes in your sketches. Compare your graphs with the graphs of your classmates, and come up with six or more different shapes that describe all 4th-degree polynomial functions.

11. Each of these is the graph of a polynomial function with leading coefficient $a = 1$ or $a = -1$.

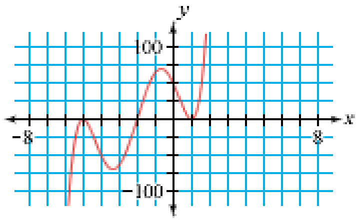
i.



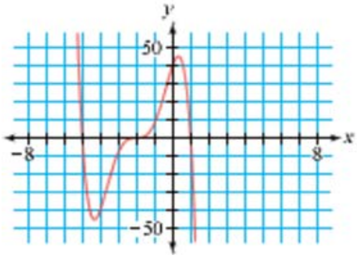
ii.



iii.



iv.



a. Write a function in factored form that will produce each graph.

b. Name the zeros of each polynomial function in 11a. If a factor is raised to the power of n, list the zero n times.

12. Consider the polynomial functions in Exercise 11.

a. What is the degree of each polynomial function?

b. How many extreme values does each graph have?

c. What is the relationship between the degree of the polynomial function and the number of extreme values?

d. Complete these statements:

i. The graph of a polynomial curve of degree n has at most __?__ x-intercepts.

ii. A polynomial function of degree n has at most __?__ real zeros.

iii. A polynomial function of degree n has __?__ complex zeros.

iv. The graph of a polynomial function of degree n has at most __?__ extreme values.

13. In the lesson you saw various possible appearances of the graph of a 3rd-degree polynomial function, and in Exercise 10 you explored possible appearances of the graph of a 4th-degree polynomial function. In Exercises 11 and 12, you found a relationship between the degree of a polynomial function and the number of zeros and extreme values. Use all the patterns you have noticed in these problems to sketch one possible graph of

a. A 5th-degree function.
b. A 6th-degree function.
c. A 7th-degree function.

Review

14. Find the roots of these quadratic equations. Express them as fractions.

a. $0 = 3x^2 - 13x - 10$
b. $0 = 6x^2 - 11x + 3$
c. List all the factors of the constant term, c, and the leading coefficient, a, for 14a and b. What do you notice about the relationship between the factors of a and c, and the roots of the functions?

15. If $3 + 5\sqrt{2}$ is a solution of a quadratic equation with rational coefficients, then what other number must also be a solution? Write an equation in general form that has these solutions.

16. Given the function $Q(x) = x^2 + 2x + 10$, find these values.

a. $Q(-3)$ **b.** $Q\left(-\frac{1}{5}\right)$ **c.** $Q\left(2 - 3\sqrt{2}\right)$ **d.** $Q(-1 + 3i)$

17. Solve this system using each method specified.

$$\begin{cases} 4x + 9y = 4 \\ 2x = 7 + 3y \end{cases}$$

a. Use an inverse matrix and matrix multiplication.
b. Write an augmented matrix, and reduce it to reduced row-echelon form.

18. **APPLICATION** According to Froude's Law, the speed at which an aquatic animal can swim is proportional to the square root of its length. (*On Growth and Form*, Sir D'Arcy Thompson, Cambridge University Press, 1961.) If a 75-foot blue whale can swim at a maximum speed of 20 knots, write a function that relates its speed to its length. How fast would a similar 60-foot-long blue whale be able to swim?

A blue whale surfaces for air in the Gulf of St. Lawrence near Les Escoumins, Québéc, Canada. The blue whale is the world's largest mammal.

LESSON 7.8

More About Finding Solutions

It isn't that they can't see the solution. It is that they can't see the problem.

G. K. CHESTERTON

You can find zeros of a quadratic function by factoring or by using the quadratic formula. How can you find the zeros of a higher-degree polynomial? Sometimes a graph will show you zeros in the form of x-intercepts, but only if they are real. And this method is often accurate only if the zeros have integer values. Fortunately, there is a method for finding exact zeros of many higher-degree polynomial functions. It is based on the procedure of long division. You first need to find one or more zeros. Then you divide your polynomial function by the factors associated with those zeros. Repeat this process until you have a polynomial function that you can find the zeros of by factoring or using the quadratic formula. Let's start with an example in which we already know several zeros. Then you'll learn a technique for finding some zeros when they're not so obvious.

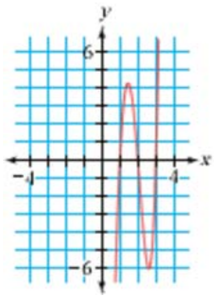
Blood is often separated into two of its factors, plasma and red blood cells.

EXAMPLE A

What are the zeros of $P(x) = x^5 - 6x^4 + 20x^3 - 60x^2 + 99x - 54$?

Solution

The graph appears to have x-intercepts at 1, 2, and 3. You can confirm that these values are zeros of the function by substituting them into $P(x)$.



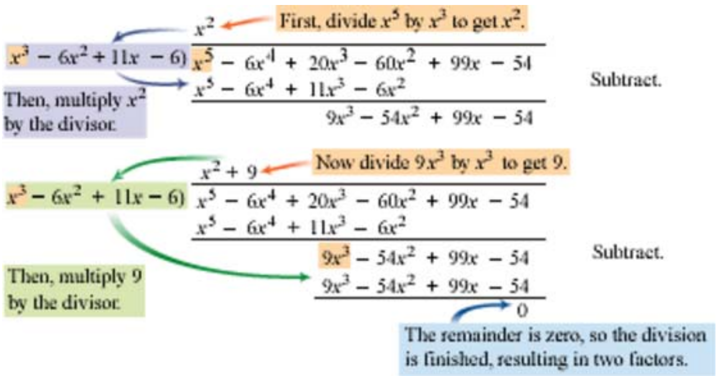
$$P(1) = (1)^5 - 6(1)^4 + 20(1)^3 - 60(1)^2 + 99(1) - 54 = 0$$
$$P(2) = (2)^5 - 6(2)^4 + 20(2)^3 - 60(2)^2 + 99(2) - 54 = 0$$
$$P(3) = (3)^5 - 6(3)^4 + 20(3)^3 - 60(3)^2 + 99(3) - 54 = 0$$

For all three values, you get $P(x) = 0$, which shows that $x = 1$, $x = 2$, and $x = 3$ are zeros. This also means that $(x - 1)$, $(x - 2)$, and $(x - 3)$ are factors of $x^5 - 6x^4 + 20x^3 - 60x^2 + 99x - 54$. None of the x-intercepts has the appearance of a repeated root, and you know that a 5th-degree polynomial function has five complex zeros, so this function must have two additional nonreal zeros.

You know that $(x - 1)$, $(x - 2)$, and $(x - 3)$ are all factors of the polynomial, so the product of these three factors, $x^3 - 6x^2 + 11x - 6$, must be a factor also. Your task is to find another factor such that

$$(x^3 - 6x^2 + 11x - 6)(\textit{factor}) = x^5 - 6x^4 + 20x^3 - 60x^2 + 99x - 54$$

You can find this factor by using long division.



Now you can rewrite the original polynomial as a product of factors:

$$x^5 - 6x^4 + 20x^3 - 60x^2 + 99x - 54 = (x^3 - 6x^2 + 11x - 6)(x^2 + 9)$$
$$= (x - 1)(x - 2)(x - 3)(x^2 + 9)$$

Now that the polynomial is in factored form, you can find the zeros. You knew three of them from the graph. The two additional zeros are contained in the factor $x^2 + 9$. What values of x make $x^2 + 9$ equal zero? If you solve the equation $x^2 + 9 = 0$, you get $x = \pm 3i$.

Therefore, the five zeros are $x = 1$, $x = 2$, $x = 3$, $x = 3i$, and $x = -3i$.

To confirm that a number is a zero, you can use the **Factor Theorem.**

Factor Theorem

$(x - r)$ is a factor of a polynomial function $P(x)$ if and only if $P(r) = 0$.

In the example, you showed that $P(1)$, $P(2)$, and $P(3)$ equal zero. You can check that $P(3i)$ and $P(-3i)$ will also equal zero.

Division of polynomials is similar to the long-division process that you may have learned in elementary school. Both the original polynomial and the divisor are written in descending order of the powers of x. If any degree is missing, insert a term with coefficient 0 as a placeholder. For example, you can write the polynomial $x^4 + 3x^2 - 5x + 8$ as

$$x^4 + 0x^3 + 3x^2 - 5x + 8$$

Insert a zero placeholder because the polynomial did not have a 3rd-degree term.

Often you won't be able to find any zeros for certain by looking at a graph. However, there is a pattern to rational numbers that might be zeros.

Rational Root Theorem

If the polynomial equation $P(x) = 0$ has rational roots, they are of the form $\frac{p}{q}$, where p is a factor of the constant term and q is a factor of the leading coefficient.

The Rational Root Theorem helps you narrow down the values that might be zeros of a polynomial function. Notice that this theorem will identify only possible *rational* roots. It won't find roots that are irrational or contain imaginary numbers.

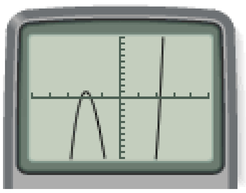
EXAMPLE B

Find the roots of this polynomial equation:

$$3x^3 + 5x^2 - 15x - 25 = 0$$

► Solution

First, graph the function $y = 3x^3 + 5x^2 - 15x - 25$ to see if there are any identifiable integer x-intercepts.



[–5, 5, 1, –10, 10, 1]

There are no integer x-intercepts, but the graph shows x-intercepts between –3 and –2, –2 and –1, and 2 and 3. Any rational root of this polynomial will be a factor of –25, the constant term, divided by a factor of 3, the leading coefficient. The factors of –25 are ± 1, ± 5, and ± 25, and the factors of 3 are ± 1 and ± 3, so the possible rational roots are ± 1, ± 5, ± 25, $\pm \frac{1}{3}$, $\pm \frac{5}{3}$, or $\pm \frac{25}{3}$. The only one of these that looks like a possibility on the graph is $-\frac{5}{3}$. Try substituting $-\frac{5}{3}$ into the original polynomial.

$$3\left(-\frac{5}{3}\right)^3 + 5\left(-\frac{5}{3}\right)^2 - 15\left(-\frac{5}{3}\right) - 25 = 0$$

Because the result is 0, you know that $-\frac{5}{3}$ is a root of the equation. If $-\frac{5}{3}$ is a root of the equation, then $\left(x+\frac{5}{3}\right)$ is a factor. Use long division to divide out this factor.

$$
\begin{array}{r|l}
 & 3x^2 - 15 \\
\hline
x+\frac{5}{3} & 3x^3 + 5x^2 - 15x - 25 \\
 & \underline{3x^3 - 5x^2} \\
 & \qquad 0 - 15x - 25 \\
 & \qquad \underline{\;\; - 15x - 25} \\
 & \qquad\qquad\qquad 0
\end{array}
$$

So $3x^3 + 5x^2 - 15x - 25 = 0$ is equivalent to $\left(x+\frac{5}{3}\right)(3x^2 - 15) = 0$. You already knew $-\frac{5}{3}$ was a root. Now solve $3x^2 - 15 = 0$.

$$3x^2 = 15$$
$$x^2 = 5$$
$$x = \pm\sqrt{5}$$

The three roots are $x = -\frac{5}{3}$, $x = \sqrt{5}$, and $x = -\sqrt{5}$. As decimal approximations, $-\frac{5}{3}$ is about -1.7, $\sqrt{5}$ is about 2.2, and $-\sqrt{5}$ is about -2.2. These values appear to be correct based on the graph.

Now that you know the roots, you could write the equation in factored form as $3\left(x+\frac{5}{3}\right)(x-\sqrt{5})(x+\sqrt{5}) = 0$ or $(3x+5)(x-\sqrt{5})(x+\sqrt{5}) = 0$. You need the coefficient of 3 to make sure you have the correct leading coefficient in general form.

When you divide a polynomial by a linear factor, such as $\left(x+\frac{5}{3}\right)$, you can use a shortcut method called **synthetic division.** Synthetic division is simply an abbreviated form of long division. Consider this division of a cubic polynomial by a linear factor:

$$\frac{6x^3 + 11x^2 - 17x - 30}{x+2}$$

Here are the procedures using both long division and synthetic division:

Long Division

$$
\begin{array}{r|l}
 & 6x^2 - 1x - 15 \\
\hline
x+2 & 6x^3 + 11x^2 - 17x - 30 \\
(-) & \underline{6x^3 + 12x^2} \\
 & \qquad -1x^2 - 17x \\
(-) & \qquad \underline{-1x^2 - 2x} \\
 & \qquad\qquad -15x - 30 \\
(-) & \qquad\qquad \underline{-15x - 30} \\
 & \qquad\qquad\qquad\quad 0
\end{array}
$$

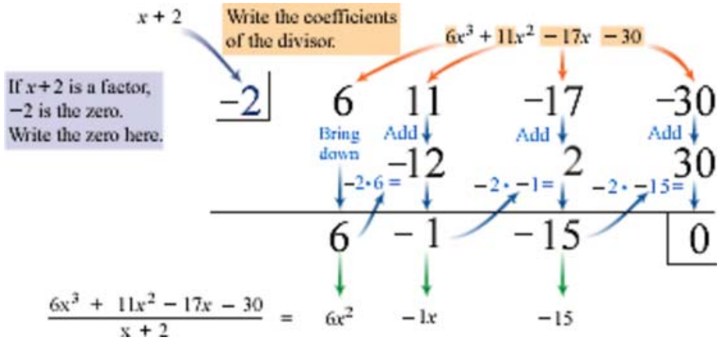
Synthetic Division

$$
\begin{array}{r|rrrr}
-2 & 6 & 11 & -17 & -30 \\
 & & -12 & 2 & 30 \\
\hline
 & 6 & -1 & -15 & \boxed{0}
\end{array}
$$

$$6x^2 - 1x - 15$$

Both methods give a quotient of $6x^2 - 1x - 15$, but synthetic division certainly looks faster. The corresponding numbers in each process are shaded. Notice that synthetic division contains all of the same information, but in a condensed form.

Here's how to do synthetic division:



The number farthest to the right in the last row of a synthetic division problem is the remainder, which in this case is 0. When the remainder in a division problem is 0, you know that the divisor is a factor. This means –2 is a zero and the polynomial $6x^3 + 11x^2 - 17x - 30$ factors into the product of the divisor and the quotient, or $(x + 2)(6x^2 - 1x - 15)$. You could now use any of the methods you've learned-simple factoring, the quadratic formula, synthetic division, or perhaps graphing-to factor the quotient even further.

EXERCISES

Practice Your Skills

1. Find the missing polynomial in each long-division problem.

a.

$$\begin{array}{r} a \qquad\qquad \\ x+5\,\overline{)\,3x^3+22x^2+38x+15} \\ \underline{3x^3+15x^2\qquad\qquad\quad} \\ 7x^2+38x+15 \\ \underline{7x^2+35x\qquad} \\ 3x+15 \\ \underline{3x+15} \\ 0 \end{array}$$

b.

$$\begin{array}{r} 2x^2+\;\;5x-\;\;3 \\ 3x-2\,\overline{)\,6x^3+11x^2-19x+6} \\ \underline{b\qquad\qquad\qquad} \\ 15x^2-19x+6 \\ \underline{15x^2-10x\qquad} \\ -9x+6 \\ \underline{-9x+6} \\ 0 \end{array}$$

2. Use the dividend, divisor, and quotient to rewrite each long-division problem in Exercise 1 as a factored product in the form $P(x) = D(x) \cdot Q(x)$. For example, $x^3 + 2x^2 + 3x - 6 = (x - 1)(x^2 + 3x + 6)$.

3. Find the missing value in each synthetic-division problem.

a.
$$\begin{array}{r|rrrr} 4 & 3 & -11 & 7 & -44 \\ & & a & 4 & 44 \\ \hline & 3 & 1 & 11 & 0 \end{array}$$

b.
$$\begin{array}{r|rrrr} -3 & 1 & 5 & -1 & -21 \\ & & -3 & -6 & 21 \\ \hline & 1 & b & -7 & 0 \end{array}$$

c.
$$\begin{array}{r|rrrr} 1.5 & 4 & -8 & c & -6 \\ & & 6 & -3 & 6 \\ \hline & 4 & -2 & 4 & 0 \end{array}$$

d.
$$\begin{array}{r|rrrr} d & 1 & 7 & 11 & -4 \\ & & -4 & -12 & 4 \\ \hline & 1 & 3 & -1 & 0 \end{array}$$

4. Use the dividend, divisor, and quotient to rewrite each synthetic-division problem in Exercise 3 as a factored product in the form $P(x) = D(x) \cdot Q(x)$.

Keymath.com More Practice Your Skills

5. Make a list of the possible rational roots of $0 = 2x^3 + 3x^2 - 32x + 15$.

Reason and Apply

6. Division often results in a remainder. In each of these problems, use the polynomial that defines P as the dividend and the polynomial that defines D as the divisor. Write the result of the division in the form $P(x) = D(x) \cdot Q(x) + R$, where R is an integer remainder. For example, $x^3 + 2x^2 + 3x - 4 = (x - 1)(x^2 + 3x + 6) + 2$.

a. $P(x) = 47$, $D(x) = 11$

b. $P(x) = 6x^4 - 5x^3 + 7x^2 - 12x + 15$, $D(x) = x - 1$

c. $P(x) = x^3 - x^2 - 10x + 16$, $D(x) = x - 2$

7. Consider the function $P(x) = 2x^3 - x^2 + 18x - 9$.

a. Verify that $3i$ is a zero.

b. Find the remaining zeros of the function P.

8. Consider the function $y = x^4 + 3x^3 - 11x^2 - 3x + 10$.

a. How many zeros does this function have?

b. Name the zeros.

c. Write the polynomial function in factored form.

9. Use your list of possible rational roots from Exercise 5 to write this function in factored form.

$$y = 2x^3 + 3x^2 - 32x + 15$$

10. When you trace the graph of a function on your calculator to find the value of an x-intercept, you often see the y-value jump from positive to negative when you pass over the zero. By using smaller windows, you can find increasingly more accurate approximations for x. This process can be automated by your calculator. The automation uses successive midpoints of each region above and below zero; it is called the **bisection method.** Approximate the x-intercepts for each equation by using the program BISECTN, then use synthetic or long division to find any nonreal zeros. [See Calculator Note 7I for the BISECTN program.]

a. $y = x^5 - x^4 - 16x + 16$

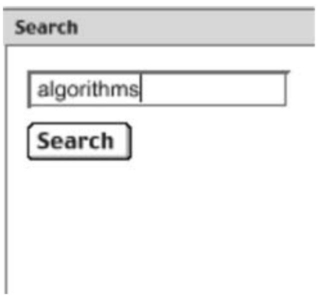
b. $y = 2x^3 + 15x^2 + 6x - 6$

c. $y = 0.2(x - 12)^5 - 6(x - 12)^3 - (x - 12)^2 + 1$

d. $y = 2x^4 + 2x^3 - 14x^2 - 9x - 12$

Technology CONNECTION

Many computer programs employ search methods that find a particular data item in a large collection of items. It would be inefficient to search for the item one piece of data at a time, so, instead, a binary search algorithm is used. First, the computer sorts the data items and checks the middle entry. If it is too low, the search algorithm will move halfway up toward the highest entry to check, or if it's too high the algorithm will check halfway toward the lowest entry. Each time, the item list is cut in half (going higher or lower) until the item being searched is reached. In a list of n items, the maximum number of times the list would have to be cut in half before finding the target is $\log_2 n + 1$.



Review

11. **APPLICATION** The relationship between the height and the diameter of a tree is approximately determined by the equation $f(x) = kx^{3/2}$, where x is the height in feet, $f(x)$ is the diameter in inches, and k is a constant that depends on the kind of tree you are measuring.

a. A 221 ft British Columbian pine is about 21 in. in diameter. Find the value of k, and use it to express diameter as a function of height.

b. Give the inverse function.

c. Find the diameter of a 300 ft British Columbian pine.

d. What would be the height of a similar pine that is 15 in. in diameter?

12. Find a polynomial function of lowest possible degree whose graph passes through the points (– 2, – 8.2), (– 1, 6.8), (0, 5), (1, – 1), (2, 1.4), and (3, 24.8).

13. **APPLICATION** Sam and Beth have started a hat business in their basement. They make baseball caps and sun hats. Let b represent the number of baseball caps, and let s represent the number of sun hats. Manufacturing demands and machinery constraints confine the production per day to the feasible region defined by

$$\begin{cases} b \geq 0 \\ s \geq 0 \\ 4s - b \leq 20 \\ 2s + b \leq 22 \\ 7b - 8s \leq 77 \end{cases}$$

They make a profit of \$2 per baseball hat and \$1 per sun hat.

a. Graph the feasible region. Give the coordinate of the vertices.

b. How many of each type of hat should they produce per day for maximum profit? (*Note:* At the end of each day, all partially made hats are recycled at no profit.) What is the maximum daily profit?

14. Write each quadratic function in general form and in factored form. Identify the vertex, y-intercept, and x-intercepts of each parabola.

a. $y = (x - 2)^2 - 16$

b. $y = 3(x + 1)^2 - 27$

c. $y = -\frac{1}{2}(x - 5)^2 + \frac{49}{2}$

d. $y = 2(x - 3)^2 + 3$

15. Solve.

a. $6x + x^2 + 5 = -4 + 4(x + 3)$

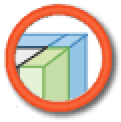
b. $7 = x(x + 3)$

c. $2x^2 - 3x + 1 = x^2 - x - 4$

CHAPTER 7 REVIEW

Polynomials can be used to represent the motion of projectiles, the areas of regions, and the volumes of boxes. When examining a set of data whose x-values form an arithmetic sequence, you can calculate the **finite differences** to find the **degree** of a polynomial that will fit the data. When you know the degree of the polynomial, you can define a system of equations to solve for the coefficients. Polynomial equations can be written in several forms. The form $a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x^1 + a_0$ is called **general form**. Quadratic equations, which are 2nd-degree polynomial equations, can also be written in **vertex form** or **factored form**, with each factor corresponding to a **root** of the equation. In a quadratic equation, you can find the roots by using the **quadratic formula**.

There are the same number of roots as the degree of the polynomial. In some cases these roots may include **imaginary** or **complex numbers.** If the coefficients of a polynomial are real, then any nonreal roots come in **conjugate pairs**. The degree of a polynomial function determines the shape of its graph. The graphs may have hills and valleys where you will find **local minimums** and **maximums**. By varying the coefficients, you can change the relative sizes of these hills and valleys.



EXERCISES

1. Factor each expression completely.

a. $2x^2 - 10x + 12$ **b.** $2x^2 + 7x + 3$ **c.** $x^3 - 10x^2 - 24x$

2. Solve each equation by setting it equal to zero and factoring.

a. $x^2 - 8x = 9$ **b.** $x^4 + 2x^3 = 15x^2$

3. Using three noncollinear points as vertices, how many different triangles can you draw? Given a choice of four points, no three of which are collinear, how many different triangles can you draw? Given a choice of five points? n points?

4. Tell whether each equation is written in general form, vertex form, or factored form. Write each equation in the other two forms, if possible.

a. $y = 2(x - 2)^2 - 16$ **b.** $y = -3(x - 5)(x + 1)$

c. $y = x^2 + 3x + 2$ **d.** $y = (x + 1)(x - 3)(x + 4)$

e. $y = 2x^2 + 5x - 6$ **f.** $y = -2 - (x + 7)^2$

5. Sketch a graph of each function. Label all zeros and the coordinates of all maximum and minimum points. (Each coordinate should be accurate to the nearest hundredth.)

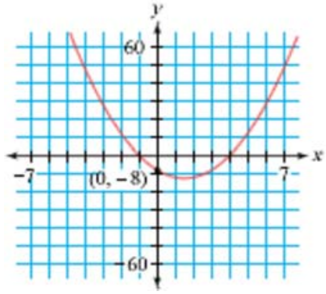
a. $y = 2(x - 2)^2 - 16$ **b.** $y = -3(x - 5)(x + 1)$

c. $y = x^2 - 3x + 2$ **d.** $y = (x + 1)(x - 3)(x + 4)$

e. $y = x^3 + 2x^2 - 19x + 20$ **f.** $y = 2x^5 - 3x^4 - 11x^3 + 14x^2 + 12x - 8$

6. Write the equation of each graph in factored form.

a.



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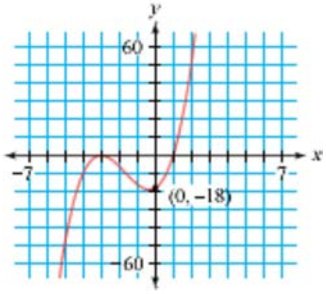

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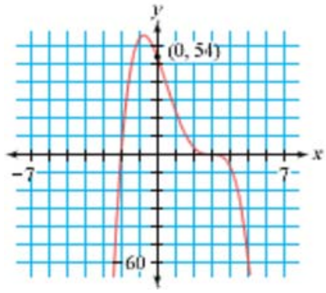

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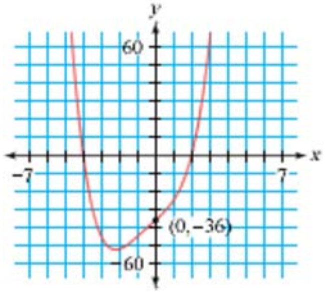
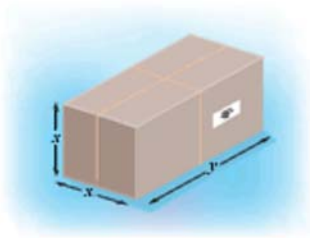


(*Hint:* One of the zeros occurs at $x = 3i$.)

7. **APPLICATION** By postal regulations, the maximum combined girth and length of a rectangular package sent by Priority Mail is 108 in. The length is the longest dimension, and the girth is the perimeter of the cross section. Find the dimensions of the package with maximum volume that can be sent through the mail. (Assume the cross section is always a square with side length x.) Making a table might be helpful.

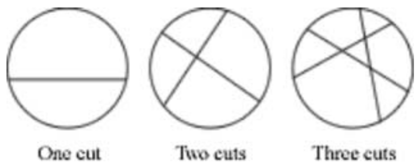


8. An object is dropped from the top of a building into a pool of water at ground level. There is a splash 6.8 s after the object is dropped. How high is the building in meters? In feet?

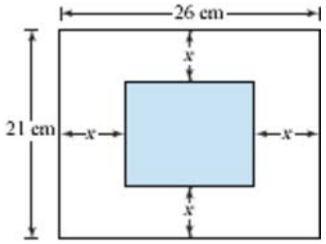
9. Consider this puzzle:

a. Write a formula relating the greatest number of pieces of a circle, y, you can obtain from x cuts.

b. Use the formula to find the maximum number of pieces with five cuts and with ten cuts.



10. This 26-by-21 cm rectangle has been divided into two regions. The width of the unshaded region is x cm, as shown.



a. Express the area of the shaded part as a function of x, and graph it.
b. Find the domain and range for this function.
c. Find the x-value that makes the two regions (shaded and unshaded) equal in area.

The Grande Arche building at La Defense in Paris, France, is in the form of a hollowed-out cube and functions as an office building, conference center, and exhibition gallery.

11. Consider the polynomial equation
$0 = 3x^4 - 20x^3 + 68x^2 - 92x - 39$.
a. List all possible rational roots.
b. Find the four roots of the equation.

12. Write each expression in the form $a + bi$.
a. $(4 - 2i)(-3 + 6i)$ **b.** $(-3 + 4i) - (3 + 13i)$ **c.** $\frac{2-i}{3-4i}$

13. Divide.
$$\frac{6x^3 + 8x^2 + x - 6}{3x - 2}$$

TAKE ANOTHER LOOK

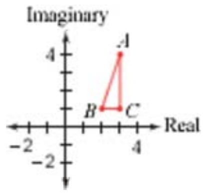
1. Use the method of finite differences to find the degree of a polynomial function that fits the data at right. What do you notice? Plot the points. What type of curve do you think might best fit the data?

x	y
0	60
1	42
2	28
3	20
4	14
5	10
6	7

2. How many points do you need to determine a line? How many points do you need to determine a parabola? Cubic curve? Quartic curve? Does the orientation of the curve-horizontal or vertical-matter? How can your conjectures be justified using the method you learned in this chapter for finding the equations of polynomial curves?

3. Choose any two complex numbers and plot them on a complex plane. Now add them and plot the resulting point. Try this with a few combinations of points. Do you see a geometric relationship between the third point and the first two? What if you subtract two complex numbers? Repeat the process: Choose two complex numbers, subtract, and plot the resulting point. Make a conjecture about the geometric relationship among the points.

4. Multiply each complex number represented by the vertices of $\triangle ABC$ by i. Plot the numbers associated with the results. Make a conjecture about the geometric meaning of $i(a + bi)$. Confirm your conjecture using other points and figures. Explore the geometric meaning of $i^2(a + bi)$. One way to do this is to multiply each complex number associated with the vertices of $\triangle ABC$ by i^2. Plot the points resulting from each multiplication. Make a conjecture for $i^n(a + bi)$. Confirm your conjecture.



Assessing What You've Learned

WRITE TEST ITEMS In this chapter you learned what complex numbers are, how to do computations with them, and how they relate to polynomial functions. Write at least two test items that assess understanding of complex numbers. Be sure to include complete solutions.

PERFORMANCE ASSESSMENT While a classmate, a friend, a family member, or a teacher observes, show how you would find all zeros of a polynomial equation given in general form, or how you would find an equation in general form given the zeros. Explain the relationship between the zeros and the graph of a function, including what happens when a particular zero occurs multiple times.

GIVE A PRESENTATION Give a presentation on how to do long division or synthetic division. Explain the advantages and the limitations of the method you have chosen, and describe a problem that could be solved using this procedure. If you like, solve the same problem using both methods, and show how they compare.